TAIT DIAGRAMS OF GRID GRAPHS

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1. Introduction

The central question in this writeup is given to students as extra credit in Jozef Przytycki's Math 1009 as a way of gauging their fluency with the necessary definitions. Answers, when presented, are given without the expectation of proof.

This writeup is meant to take an elementary approach to solving the problem, requiring only basic group theory and some knowledge of links and graphs. Thank you to Professor Lewis for helping me to reformulate my question, and to Professor Przytycki for introducing me to the problem (it's actually the first I remember receiving from him!).

Question
How many components does the Tait Diagram of the Grid Graph $G_{m,n}$ have?

That is, what is $\#C(D(G_{m,n}))$?

Let's begin by defining all the relevant terms.

2. Tait Diagrams & Grid Graphs

2.1. **Tait Diagram.** The Tait Diagram of a planar graph G is a link diagram associated to G. It is constructed in the following way:

To each edge of the graph G, we associate a crossing of our link diagram, $D(G)$. Regarding an edge of G as a North-South longitude, we place an overcrossing along the NE-SW axis and an undercrossing along the NW-SE axis as in Figure 2.1:

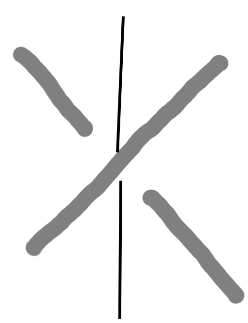


Figure 2.1. Crossing of Tait Diagram Along Edge of Graph

With every crossing drawn along the edges of G, we connect each understrand of a crossing to the closest overstrand. Connected arcs will lie along edges of G with a common vertex, and their connection will never cross over an edge of G. See Figure 2.2 for an example. As described, this construction will always produce an alternating link diagram.

For more on Tait Diagrams of planar graphs, see the remark at the end, which illustrates the algorithm by which we can produce any link diagram.

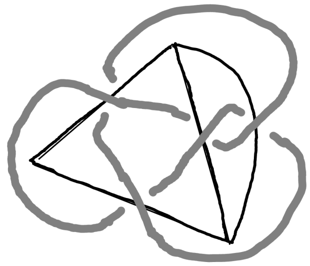


FIGURE 2.2. Tait Diagram of (Unsigned) Graph

2.2. **Grid Graphs and Components of Link Diagrams.** We give the definition of Grid Graph and then the definition of link components.

Definition. A grid graph $G_{m,n}$ is an $(m+1) \times (n+1)$ lattice-graph; that is, the graph Cartesian Product $P_{m+1} \times P_{n+1}$ of the path graphs of $m+1$ and $n+1$ vertices. In our naming convention, the grid graph $G_{m,n}$ has m rows of boxes (circulants C_4) and n columns of boxes. [1]

See Figure 2.3.

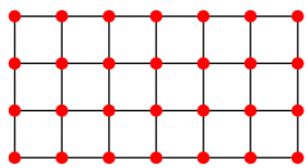


FIGURE 2.3. Grid Graph $G_{3,6}$

Definition (Components of Link Diagram)**.** Intuitively, a component of a link diagram is one of finitely many images of circles in the plane which comprise said diagram. In Figure 2.4, we have a two-component link. Note that in particular, knots are links of 1 components.

Formally, for a link L defined by the embedding $l : \bigsqcup^n S^1 \hookrightarrow S^3$, the link diagram of L is identified by the image of an immersion p which projects the image of l to $\mathbb{R}^2$ with finitely many double points. A component of a link diagram is the image of a copy of S^1 under $p \circ l$. The number of link components is n.

In our solution, we also refer to the connected components of graphs, and though I will use $C(*)$ to denote the components of both graphs and links, the referenced object should always be clear through context.

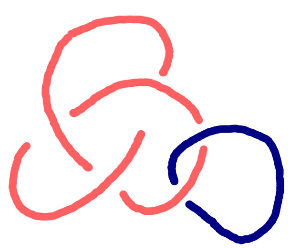


FIGURE 2.4. A link diagram with components in red and blue

With that last definition, we have all the necessary machinery to understand the question asked:

Question
How many components does the Tait Diagram of the Grid Graph $G_{m,n}$ have?

That is, what is $\#C(D(G_{m,n}))$?

3. Solution

To find $\#C(D(G_{m,n}))$, we build the following string of equalities:

$$\#C(D(G_{m,n})) = \#C(G_{x,y_n}) = \frac{l(xy_n)}{2} = \gcd(m+1, n+1).$$

We restate, define, and prove the individual equalities from left to right, building a bridge from Unknown to Known.

3.1. Tait Tangles and Permutations. ??

To begin, let's define an intermediate object: the "Tait Tangle."

Definition (Tait Tangle)**.** Let $\tau(G_{m,n})$ be the $2m+2$-tangle defined by forming the crossings of the Tait Diagram of $G_{m,n}$ and closing all but the $2m+2$ arcs on the left- and right-hand sides of the diagram as in Figure 3.1.

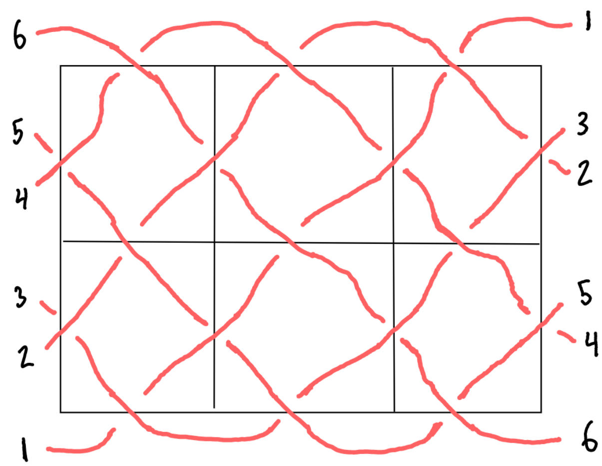


Figure 3.1. Tait Tangle of $G_{2,3}$

Let's label the left-hand-side arcs of $\tau(G_{m,n})$ from bottom to top by $1 \leq i \leq 2m+2$ and mark the output arcs on the right-hand side again as in Figure 3.1.

Our goal is to analyze how closing the Tait Tangle to produce the Tait Diagram is an equivalence relation on the arcs. Thus, counting the number of equivalence classes of the $2m+2$ arcs of the Tait Tangle after the pairwise identification of LHS and RHS arcs is precisely counting the number of components of $D(G_{m,n})$.

Clear from the picture of the Tait Tangle and the definition of the Tait Diagram is that the aforementioned closure operation which takes $\tau(G_{m,n}) \to D(G_{m,n})$ identifies adjacent pairs of strands from bottom to top. In particular, the LHS strands are always identified as

$$[1] = [2], [3] = [4], \dots, [2m+1] = [2m+2].$$

See Figure 3.2.

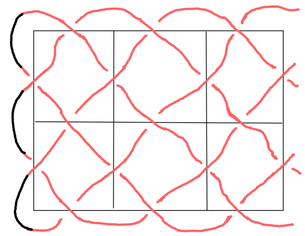


FIGURE 3.2. Left-hand side closure of Tait Tangle of Grid Graph $G_{2,3}$

Thus, having only taken into account the identifications of the LHS, we produce an immediate upper bound on the number of components of $D(G_{m,n})$:

$$\#C(D(G_{m,n})) \leq m + 1.$$

Further, we note that

(1) This maximum is achieved for some m, n: e.g. $\#C(G_{3,3}) = 4$ and
(2) the inequality is not equality for all m, n: e.g. $\#C(G_{3,4}) = 1$

This second fact should be apparent by considering the equivalence relations further imposed on the $2m + 2$ arcs by the right-hand side of $\tau(G_{m,n})$. For instance, notice that the identifications of arcs on the RHS of $\tau(G_{2,3})$ in $D(G_{2,3})$ reduce $m + 1$ equivalence classes to a single equivalence class in Figure 3.3.

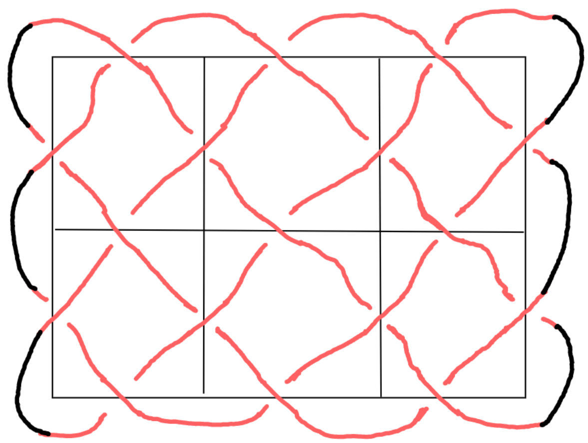


FIGURE 3.3. Caption

Seeing that RHS closure of the Tait Tangle can reduce the number of components of the Tait Diagram , we need a way need a way to predict the ordering of the output arcs on the RHS for a given n. Doing so will enable us to count further identifications of our distinct arcs.

Thankfully, the second advantage of defining the Tait Tangle is that the addition of a column to $G_{m,n}$ permutes the arcs on the RHS of the Tait Tangle in a regular way. Thus, the problem is totally combinatorial. If we can figure out how the RHS arcs permute from $\tau(G_{m,n})$ to $\tau(G_{m,n+1})$, we can start with the output arcs of $\tau(G_{m,1})$ and predict the ordering of the output arcs for any n. Given that ordering, the RHS closure operation identifies arcs in pairs, possibly reducing the number of distinct equivalence classes.

To show how the addition of a column corresponds to a fixed permutation on the RHS arcs, one can notice the ordering of the arcs on the RHS of $\tau(G_{2,3})$, $\tau(G_{2,4})$, and $\tau(G_{2,5})$ in Figure 3.4.

We describe this permutation and formalize our observations in the construction of the graph H_{x,y_n} associated to $D(G_{m,n})$ in the following section.

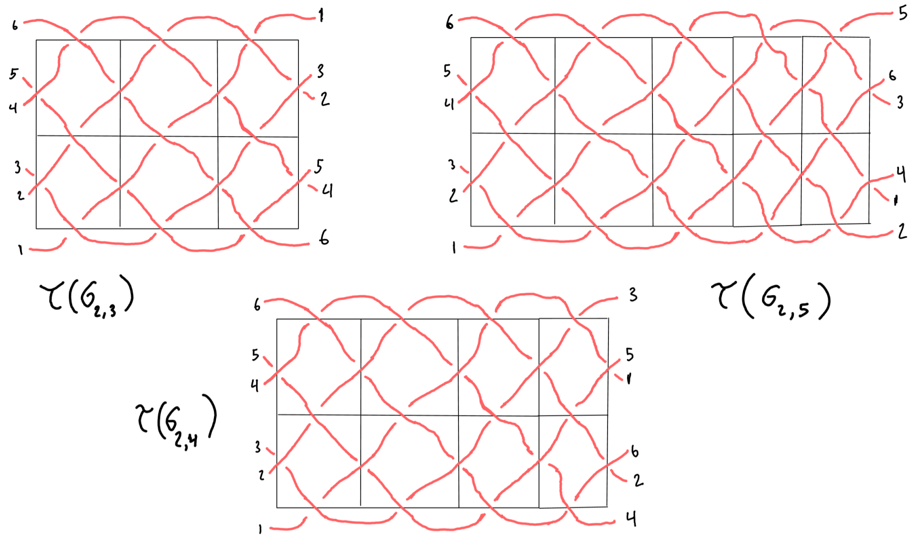


FIGURE 3.4. Each additional column of the grid graph permutes the RHS arcs

3.2. The Graph H_{x,y_n}.

For a given m, let

$$x = (1\ 2)(3\ 4)\ldots(2m+1\ 2m+2) \in S_{2m+2}$$

The entries in each transposition of x are the labellings of the arcs of the Tait Tangle which are identified on the LHS of the Tait Diagram. That is x_i, x_i' appear in a transposition of x iff the arcs x_i and x_i' are in the same equivalence class after the closure operation is performed on the LHS of the Tait Tangle. This is simply a restatement of the observations of the previous section. In fact, our remark that the LHS closure yielded an upper bound of $m+1$ components is evident in there being $m+1$ cycles of x.

For a similar description of the RHS arcs of $\tau(G_{m,n})$, we have the fixed-point-free involution

$$y_n = \sigma^{n+1} x \sigma^{-n-1}$$

where

$$\sigma = (1\ 3\ 5\ \ldots\ 2m+1\ 2m+2\ 2m\ \ldots\ 6\ 4\ 2).$$

Again, transpositions of y_n have as entries arcs which are identified on the RHS of the Tait Diagram.

This description of y_n demonstrates the correspondence of additional columns of $G_{m,n}$ with permutations of RHS arcs: an additional column corresponds to conjugation by σ.

For an example, take $\tau(G_{2,3})$, as shown in Figure 3.4.

The RHS arcs of $\tau(G_{2,3})$ are identified by the transpositions of

$$\begin{aligned} y_3 &= \sigma^4 x \sigma^{-1} \\ &= (1\ 3\ 5\ 6\ 4\ 2)^4 (1\ 2)(3\ 4)(5\ 6)(2\ 4\ 6\ 5\ 3\ 1)^4 \\ &= (6\ 4)(5\ 2)(3\ 1) \end{aligned}$$

and the RHS arcs of $\tau(G_{2,4})$ by

$$
\begin{aligned}
y_4 &= \sigma^5 x \sigma^{-5} \\
&= \sigma y_3 \sigma^{-1} \\
&= (1\ 3\ 5\ 6\ 4\ 2)(6\ 4)(5\ 2)(3\ 1)(2\ 4\ 6\ 5\ 3\ 1) \\
&= (4\ 2)(6\ 1)(5\ 3).
\end{aligned}
$$

We will now use the fixed-point-free involutions x, y_n to define a graph which captures information about the link components of $D(G_{m,n})$

Definition. Let H_{x,y_n} be the graph on $2m+2$ labelled vertices with an edge between vertices which appear in the same transposition of either x or y_n.

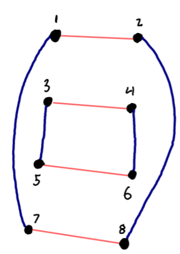
For example, for grid graph $G_{3,1}$, we have

$$x = (1\ 2)(3\ 4)(5\ 6)(7\ 8),$$

and

$$
\begin{aligned}
y_1 &= (1\ 3\ 5\ 7\ 8\ 6\ 4\ 2)^2(1\ 2)(3\ 4)(5\ 6)(7\ 8)(2\ 4\ 6\ 8\ 7\ 5\ 3\ 1)^2 \\
&= (1\ 7)(2\ 8)(3\ 5)(4\ 6).
\end{aligned}
$$

Then the graph H_{x,y_1} corresponding to $D(G_{3,1})$ is



Red edges correspond to the identification of arcs on the LHS of $\tau(G_{m,n})$, encoded by the transpositions of x. Blue edges correspond to identifications of arcs on the RHS, encoded by transpositions of y_n.

3.3. **Equality 1.** By analyzing definition of H_{x,y_n}, our first equality falls out.

Claim.

$$\#C(H_{x,y_n}) = \#C(D(G_{m,n})).$$

By definition, there's a bijection between

$$\{\text{vertices of } H_{x,y_n}\} \leftrightarrow \{\text{arcs of } \tau(G_{m,n})\},$$

a bijection between

$$\{\text{edges of } H_{x,y_n}\} \leftrightarrow \{\text{identification of arcs of } \tau(G_{m,n}) \text{ in } D(G_{m,n})\},$$

and thusly between

$$\{\text{components of } H_{x,y_n}\} \leftrightarrow \{\text{components of } D(G_{m,n})\}.$$

3.4. **Equality 2.** Our second equality relates the number of connected components of the graph H_{x,y_n} with the number of disjoint cycles in the cyclic decomposition of the product of the fixed-point-free involutions xy_n.

Proposition 1. For every connected component of H_{x,y_n}, there are two disjoint cycles in the cycle decomposition of xy_n.

That is,

$$\#C(H_{x,y_n}) = \frac{l(xy_n)}{2},$$

where $l(\sigma)$ denotes the number of disjoint cycles in the cycle decomposition of the permutation σ.

Proof. Let $C \subset H_{x,y_n}$ be a connected component of H_{x,y_n}. We note that C is a cycle graph, though we will not appeal to this fact in our argument.

Let $x^C, y_n^C \in S_{2m+2}$ be the subpermutations of x, y_n with transpositions containing all $j \in V(C)$.

That is,

$$x^C = (x_1\ x_2)(x_3\ x_4)\dots(x_{2k-1}\ x_{2k}),$$

$$y_n^C = (y_1\ y_2)(y_3\ y_4)\dots(y_{2k-1}\ y_{2k})$$

where $V(C) \subset \{x_i\}_{i=1}^{2k}$, $V(C) \subset \{y_i\}_{i=1}^{2k}$.

We argue that $V(C) = \{x_i\}_{i=1}^{2k} = \{y_i\}_{i=1}^{2k}$ by showing the reverse inclusion.

To show that $V(C) \supset \{x_i\}_{i=1}^{2k}$, we ask if it were possible that an entry $x_i \in V(C)$ which appears in a transposition $(x_i\ x_i')$ of x^C could have $x_i' \notin V(C)$.

By definition of H_{x,y_n}, the vertex labelled x_i has an edge connecting it to x_i', meaning x_i' is part of the same connected component as x_i in the graph H_{x,y_n} and thus $x_i' \in V(C)$. (The same argument holds for an entry of y_i, which shows that $V(C) = \{x_i\} = \{y_i\}$.)

Now that we are happy with the definition of x^C, y_n^C, we prove the claim.

We write

$$x^C = (x_1\ x_2)(x_3\ x_4)\dots(x_{2k-1}\ x_{2k})$$

$$y_n^C = (y_1\ y_2)(y_3\ y_4)\dots(y_{2k-1}\ y_{2k})$$

and compare the cycles of x_n, y_n^C.

There are two cases:

- Repeated Cycle: If for any $i, j \in [2k]$, $(x_i\ x_{i+1}) = (y_j\ y_{j+1})$, then

  $$x^C = y_n^C = (x_i\ x_{i+1}) = (y_j\ y_{j+1}).$$

  In this case, C is a length-two cycle graph, and

  $$x_n y_n^C = (x_i)(x_{i+1}).$$

  Thus, consistent with what we intended to show, $l(x^C y_n^C) = 2$. The connected component C is in $1:2$ correspondence with the number of disjoint cycles in the product $x^C y_n^C$.

- Distinct Cycles: If x^C, y_n^C do not share a transposition, then for any transposition $(y_i\ y_{i+1})$ of y_n^C

$$x^C(y_n^C(y_i)) = x^C(y_{i+1}) \text{ and } x^C(y_n^C(y_{i+1})) = x^C(y_i),$$

but

$$x^C(y_{i+1}) \neq x^C(y_i)$$

since x^C is injective.

Then, we note that since x^C, y_n^C do not share a cycle,

$$(x^C \circ y_n^C)^n(y_1) \neq (x^C \circ y_n^C)^i(y_1)$$

for all $i \neq n < k$. The argument relies only on the facts that 1) x^C and y_n^C do not share any cycles, and that 2) x^C, y_n^C are fixed-point-free involutions.

Thus, the product $x^C y_n^C$ has a cycle decomposition into the orbit of y_1 and the orbit of y_2 under $(x^C \circ y_n^C)$; i.e.

$$x^C y_n^C = \Big(y_1\ (x^C \circ y_n^C)(y_1)\ \dots\ (x^C \circ y_n^C)^{k-1}(y_1)\Big)\Big(y_2\ (x^C \circ y_n^C)(y_2)\ \dots\ (x^C \circ y_n^C)^{k-1}(y_2)\Big).$$

We conclude that the connected component C is again in $1:2$ correspondence with the number of disjoint cycles in the product $x^C y_n^C$.

Summing over the connected components in G_{x,y_n}, we retrieve our result:

$$\#C(H_{x,y_n}) = \frac{l(xy_n)}{2}.$$

□

3.5. **Equality 3.** Thus, we have only one final equality to prove to complete our proof.

$$\#C(D(G_{m,n})) = \#C(G_{x,y_n}) = \frac{l(xy_n)}{2} = \gcd(m+1, n+1)$$

Proposition 2.

$$\frac{l(xy_n)}{2} = \gcd(m+1, n+1)$$

Proof. We recall that $x = (1\ 2)(3\ 4)\dots(2m+1\ 2m+2)$ and

$$y_n = \sigma^{n+1} x \sigma^{-n-1}$$

where

$$\sigma = (1\ 3\ 5\ \dots\ 2m+1\ 2m+2\ 2m\ \dots\ 6\ 4\ 2).$$

We analyze the product $xy_n = x\sigma^{n+1}x\sigma^{-n-1}$.

First, we observe that $x\sigma x = \sigma^{-1}$:

$$\begin{aligned} x\sigma x &= (1\ 2)(3\ 4)\dots(2m+1\ 2m+2)(1\ 3\ 5\ \dots 2m+1\ 2m+2\ 2m \dots\ 4\ 2)(1\ 2)(3\ 4)\dots(2m+1\ 2m+2) \\ &= (x(1)\ x(3)\ x(5)\ \dots x(2m+1)\ x(2m+2)\ x(2m)\dots\ x(4)\ x(2)) \\ &= (2\ 4\ 6\ \dots 2m+2\ 2m+1\ 2m-1 \dots\ 3\ 1) \\ &= \sigma^{-1} \end{aligned}$$

Then,

$$(x\sigma x)^n = (\sigma^{-1})^n = \sigma^{-n}.$$

Therefore

$$\begin{aligned} xy_n &= x\sigma^{n+1}x\sigma^{-n-1} \\ &= (x\sigma x)^{n+1}\sigma^{-n-1} \\ &= \sigma^{-n-1}\sigma^{-n-1} \\ &= \sigma^{-2n-2}. \end{aligned}$$

Since we know that $l(xy_n) = l(\sigma^{-2n-2}) = 2\#C(H_{x,y_n})$, we need only find the number of disjoint cycles in the cycle decomposition of σ^{-2n-2}. For that, we have the following lemma.

Lemma. Let σ be an m cycle. Then $l(\sigma^k) = \gcd(m,k)$.

Proof. We write

$$\sigma = (s_0\ s_1\ s_2 \dots s_{m-1})$$

and notice that $\sigma^k(s_i) = s_{[i+k]_m}$ where by $[n]_m$ we denote the canonical representative of n in $\mathbb{Z}/m\mathbb{Z}$.

For any entry s_i of σ, the (finite length) disjoint cycle in the cycle decomposition of σ^k containing s_i is of the form

$$(s_i\ \sigma^k(s_i)\ \sigma^{2k}(s_i)\ \dots) = (s_i\ s_{[i+k]_m}\ s_{[i+2k]_m} \dots).$$

We naturally identify this cycle with a $\langle [k] \rangle$-cosets of $\mathbb{Z}/m\mathbb{Z}$,

$$i + \langle [k] \rangle = \{i, i+k, i+2k \dots\}.$$

Thus, to count the number of disjoint cycles of σ^k, we need to count the number of such $\langle [k] \rangle$-cosets of $\mathbb{Z}/m\mathbb{Z}$; that is, find $[\mathbb{Z}/m\mathbb{Z} : \langle [k] \rangle]$.

By Lagrange's Theorem,

$$\begin{aligned} [\mathbb{Z}/m\mathbb{Z} : \langle [k] \rangle] &= \frac{|\mathbb{Z}/m\mathbb{Z}|}{|\langle [k] \rangle|} \\ &= \frac{m}{|\langle [k] \rangle|.} \end{aligned}$$

What is $|\langle [k] \rangle|$? Well, the unique elements of $\langle [k] \rangle$ look like

$$\{[k], [2k], [3k], \dots, [lk]\}$$

where l is the least integer such that $m \mid lk$. Thus, $l = \mathrm{lcm}(m,k)$ by definition, and $|\langle [k] \rangle| = \frac{\mathrm{lcm}(m,k)}{k}$.

We compute

$$\begin{aligned} [\mathbb{Z}/m\mathbb{Z} : \langle [k] \rangle] &= \frac{m}{|\langle [k] \rangle|} \\ &= \frac{m}{\mathrm{lcm}(m,k)/k} \\ &= \frac{mk}{\mathrm{lcm}(m,k)} \\ &= \gcd(m,k). \end{aligned}$$

□

Applying this lemma to our original statement, we have that σ is a $2m+2$-cycle, so

$$l(\sigma^{-2n-2}) = l(\sigma^{2n+2}) = \gcd(2m+2, 2n+2) = 2\gcd(m+1, n+1).$$

So we have that

$$l(xy_n) = l(\sigma^{-2n-2}) = 2\gcd(m+1, n+1).$$

This proves our final equality in the chain

$$\#C(D(G_{m,n})) = \#C(G_{x,y_n}) = \frac{l(xy_n)}{2} = \gcd(m+1, n+1),$$

and in doing so, proves that

$$\#C(D(G_{m,n})) = \gcd(m+1, n+1).$$

□

Remark (Nonalternating Tait Diagrams)**.** *Expanding upon the method of constructing link diagrams from planar graphs, we can in fact produce all link diagrams via a similar algorithm to the one given before. This detail is not particularly useful for answering the main question of this write-up, but good to know.*

Let G be a planar graph and let $\epsilon : E(G) \to \{-1, 1\}$ a function. We call (G, ϵ) a signed graph, which can be regarded as the planar graph G with edges decorated by either $(+)$ or $(-)$.

To form the Tait Diagram of a signed graph, we again associate a crossing of $D(G)$ with each edge of the graph G.

- *To each $(+)$-decorated edge, we associate a crossing of $D(G)$ as before.*
- *To each $(-)$-decorated edge, we form the opposite crossing; i.e. our undercrossing is placed along the NE-SW axis and our overcrossing along the NW-SE axis.*
- *The arcs of these crossings are connected to neighboring arcs as before, though this construction may result in non-alternating diagrams.*

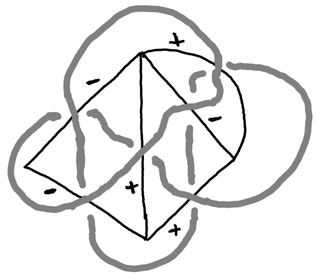


Figure 3.5. Tait Diagram of Signed Planar Graph

This construction actually yields a one-to-one correspondence

$$\{\textit{Signed Planar Graphs} \leftrightarrow \{\textit{Link Diagrams}\}.$$

*In particular, there is an also an explicit algorithm for constructing a signed planar graph from an link diagram. See [**prz**] for more.*

References

[1] Eric W. Weisstein. *Grid Graph*. URL: https://mathworld.wolfram.com/GridGraph.html.