USING FIBONACCI NUMBERS AND CHEBYSHEV POLYNOMIALS TO EXPRESS FOX COLORING GROUPS AND ALEXANDER-BURAU-FOX MODULES OF DIAGRAMS OF WHEEL GRAPHS

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Abstract. In this paper we compute the Reduced Fox Coloring Group of the diagrams of Wheel Graphs which can also be represented at the closure of the braids $(\sigma_1\sigma_2^{-1})^n$. In doing so, we utilize Fibonacci numbers and their properties.

Following this, we generalize our result to compute the Alexander-Burau-Fox Module over the ring $\mathbb{Z}[t^{\pm 1}]$ for the same class of links. In our computation, Chebyshev polynomials function as a generalization of Fibonacci Numbers.

Contents

1. Introduction

In this section we introduce the Tait Diagram of a plane graph. We then recall the definition of Fox n-colorings and of the Fox Coloring Group of a link (the universal object for Fox Colorings). Then, we formulate our main result about the structure of Fox Coloring Groups.

In the second section, we prove our main result about Fox Colorings by working with the matrix of relations for the Fox Group using the properties of Fibonacci numbers.

In the third section, we recall the definition of the Alexander-Burau-Fox Module over the ring $\mathbb{Z}[t^{\pm 1}]$. This module is a generalization of our Fox Coloring Group. We describe the

2020 *Mathematics Subject Classification.* Primary: 57K10 Secondary: 57M12, 11B39.

Key words and phrases. Determinants of links, Fox coloring, Alexander-Burau-Fox module, Burau representation, Fibonacci numbers, Chebyshev polynomials, knots and links.

structure of the ABF Module for the same family of links considered in section one and two, and express it as the sum of two cyclic modules.

In the fourth section, we mention the relation with Plan's Theorem on branch covers of links. We also relate our results to those of Minkus and Mulazzani-Vesnin.

We now present some basic definitions.

Definition 1.1. *For a plane graph G, we associate an alternating link diagram as follows:*

(1) *Every edge is replaced by a crossing as illustrated in Figure 1.*
(2) *We connect the "loose endpoints" of the crossing along the edges; compare Figure 3.*

FIGURE 1. Crossing from an edge

Definition 1.2. *We define a Fox n-coloring of a diagram D to be a function*

$$f : arcs(D) \to \mathbb{Z}_n$$

such that every arc is "colored" by an element of $\mathbb{Z}_n$ with the following condition: for every crossing with arcs a, b, and c, $2b - a - c \equiv 0 \mod n$ for overarc b. That is, each crossing has the relation

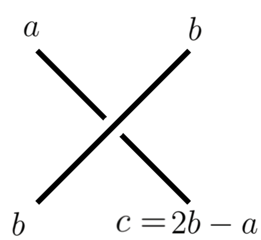


FIGURE 2. Coloring of a crossing

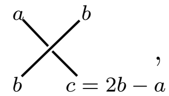
The n-colorings of a diagram D form a group denoted by $Col_n(D)$.

If $f(a_i) = f(a_j)$ for all $a_i, a_j \in arcs(D)$, we call f a trivial coloring. These trivial colorings form the subgroup $Col_n^{trivial}(D) \cong \mathbb{Z}_n \in Col_n(D)$ and the quotient $Col_n(D)/Col_n^{trivial}(D)$ is called the Reduced Group of Fox n-colorings denoted by $Col_n^{red}(D)$.

Definition 1.3. *An arc is the part of a diagram from undercrossing to undercrossing. We also include in our definition components without a crossing.*

The number of arcs is equal to the number of crossing plus the number of trivial components of the diagram.

Definition 1.4. *The group $Col(D)$ is the abelian group whose generators are indexed by the arcs of D. The set of arcs is denoted by $arcs(D)$ and the set of generators is denoted by $Arcs(D)$. The relations at each crossing of D are given by $2b - a - c = 0$ where $a, b, c \in Arcs(D)$. That is,*

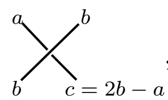
$$Col(D) = \{Arcs(D) \mid$$ 

$$, \quad \text{where } a, b, c \in Arcs(D)\}$$

The Fox Coloring Group can be also defined for tangles in a similar way. In particular for braids(for n braids treated as n-tangles), the group $Col(T)$ is freely generated by the top arcs of the braid. We will use this later in the proof of Theorem 1.6.

Definition 1.5. *Let $Col^{trivial}(D) \leq Col(D)$ be the infinite cyclic subgroup generated by the element $\sum_{a_i \in Arcs(D)} a_i$. This subgroup is isomorphic to $\mathbb{Z}$ and is called the group of trivial colorings of D.*

The quotient group $\dfrac{Col(D)}{Col^{trivial}(D)}$ is called the ***reduced group of Fox colorings***.[1] *We denote it by $Col^{red}(D)$. That is,*

$$Col^{red}(D) = \{Arcs(D) \mid$$ 

$$, \quad \sum_{a_i \in Arcs(D)} a_i = 0\},$$

where the sum is taken over all arcs of D.

More information about Fox Colorings can be found in [Prz1, PBIMW].

We are now ready to formulate the main theorem of the second section, expressing the Reduced Fox Coloring Group using Fibonacci numbers.

Theorem 1.6. *Let F_k be the Fibonacci sequence defined by*

$$F_0 = 0, \;\; F_1 = 1 \text{ and } F_{k+2} = F_{k+1} + F_k.$$

Let D_n be the closure of the braid $(\sigma_1 \sigma_2^{-1})^n$, that is, $D_n = D(W_n)$ as in Figure 3. Then

$$Col^{red}(D_n) = \begin{cases} \mathbb{Z}_{F_{n-1}+F_{n+1}} \oplus \mathbb{Z}_{F_{n-1}+F_{n+1}} & \text{when } n \text{ is odd,} \\ \mathbb{Z}_{5F_n} \oplus \mathbb{Z}_{F_n} & \text{when } n \text{ is even} \end{cases}$$

In particular, for $n = 2, 3, 4, 5, 6, 7,$ we have $\mathbb{Z}_5$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_{15} \oplus \mathbb{Z}_3$, $\mathbb{Z}_{11} \oplus \mathbb{Z}_{11}$, $\mathbb{Z}_{40} \oplus \mathbb{Z}_8$, and $\mathbb{Z}_{29} \oplus \mathbb{Z}_{29}$, respectively.

2. Fibonacci numbers and reduced coloring group of wheel graphs, $Col(D)$

After a few preliminaries, we turn to a proof of Theorem 1.6.

The Tait diagram of a planar graph has a determinant equal to the number of spanning trees of the graph. For wheel graphs in particular, the number of spanning trees was computed in [Sed, Mye]. For our purposes, the determinant is equal to the order of the Reduced Fox Coloring Group. If the determinant is zero, which may only occur in the case of links, then it may result in infinite order.

[1]The group $Col^{red}(D)$ can be interpreted as the first homology of the double branch covering of S^3 branched along D; see [Prz1]. The group $Col^{red}(D)$ can also be computed by the Goeritz matrix of the diagram. Using this approach, the determinant of D_n is computed in [Prz2].

In the second section of the paper, we obtain concise formulas, using Fibonacci numbers, for the Fox-Coloring Group of diagrams obtained from wheel graphs, which can also be expressed as the closure of 3-braids of the form $(\sigma_1\sigma_2^{-1})^n$ where n is the number of spokes in the corresponding wheel graph; compare Figure 3.

FIGURE 3. Wheel graph W_7 and its Tait diagram, $D_7 = D(W_7)$ representing the closure of the braid $(\sigma_1\sigma_2^{-1})^7$.

The proof is comprised of several propositions and lemmas.

Proposition 2.1. *Consider an arbitrary 3-braid B and label the top arcs by a, b and c and the bottom arcs by a', b', and c', which are uniquely defined by a, b and c.*

(1) *In the Fox Coloring Group of this 3-braid, our arcs satisfy the equation $(a'-a)-(b'-b)+(c'-c)=0$.*
(2) *Recall that a, b and c form a basis of $Col(B)$ and we can change the basis to b, $b-a$ and $b-c$. Then $a'-a$, $b'-b$, and $c'-c$ are linear combinations of $b-a$ and $b-c$.*

Proof. The proposition is well-known and can be proven by the induction on the number of crossings, see e.g [DJP]. □

To simplify notation, let $x = b-a$ and $y = b-c$. Let $\hat{B}$ be the closure of B, then $Col(\hat{B}) = \{b, x, y \mid a'-a, c'-c\}$. Notice that by Proposition 2.1, $a'-a$ and $c'-c$ is a linear combination of x and y.[2] Let $a'-a = P_B = P_B(x,y) = P_B^x \cdot x + P_B^y \cdot y$ and $c'-c = Q_B = Q_B(x,y) = Q_B^x \cdot x + Q_B^y \cdot y$.

The matrix of relations for $Col^{red}(\hat{B})$ is a 2×2 matrix

$$\begin{bmatrix} P_B^x & P_B^y \\ Q_B^x & Q_B^y \end{bmatrix}.$$

Now we work with $B = (\sigma_1\sigma_2^{-1})^n$ and $\hat{B} = D(W_n)$. Therefore, we use notation $P_B = P_n$ and $Q_B = Q_n$.

The following lemmas are illustrated by Figure 4.

[2]The fact that x and y generate Col^{red} can be easily proved by linear induction. There is a general fact, see e.g. [DJP] that if we consider a general n-tangle, T with boundary points denoted by $x_1, x_2, ..., x_{2n}$ (variable indexed by endpoints) then the relation $\sum_{i=1}^{2n}(-1)^i x_i = 0$ holds in $Col(T)$. In particular elements $x_i - x_{i+1}$ generate $Col^{red}(T)$ group.

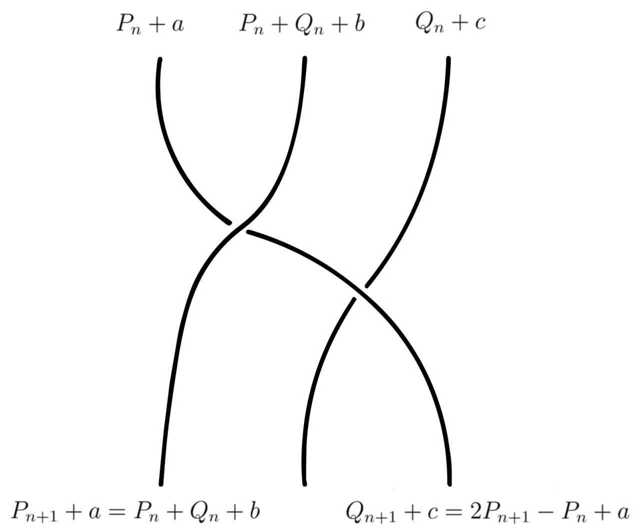


FIGURE 4. Labeling part of $(\sigma_1\sigma_2^{-1})^n$

Lemma 2.2. (1) $P_{n+1} = P_n + Q_n + b - a = P_n + Q_n + x$; *thus* $Q_n = P_{n+1} - P_n - x$.

(2) $Q_{n+1} = 2P_{n+1} - P_n + a - c = 2P_{n+1} - P_n + y - x \stackrel{(1)}{=} P_{n+1} + Q_n + y$; *thus* $P_{n+1} = Q_{n+1} - Q_n - y$.

Thus we can deduce recursive formulas for P_n and Q_n.

Lemma 2.3. (Rec1) $P_{n+2} \stackrel{(1)}{=} P_{n+1} + Q_{n+1} + x \stackrel{(2)}{=} 3P_{n+1} - P_n + y$.
(Rec2) $Q_{n+2} \stackrel{(2)}{=} P_{n+2} + Q_{n+1} + y \stackrel{(1)}{=} P_{n+1} + 2Q_{n+1} + x + y \stackrel{(2)}{=} 3Q_{n+1} - Q_n + x$.

Then using the notation $P_n = P_n^x \cdot x + P_n^y \cdot y$ and $Q_n = Q_n^x \cdot x + Q_n^y \cdot y$ we recognize P_n^x and Q_n^y as Chebyshev polynomials as follows:

Lemma 2.4. (I) $P_{n+2}^x = 3P_{n+1}^x - P_n^x$ *with* $P_0^x = 0,\ P_1^x = 1,\ P_2^x = 3,\ \ldots$
Thus $P_{n+1}^x = S_n(3)$ *where* $S_n(z)$ *denotes the Chebyshev polynomial of the second kind. That is* $S_0(z) = 1, S_1(z) = z$, *and* $S_{n+2}(z) = zS_{n+1}(z) - S_n(z)$. *Compare Section 3.1.*
$Q_{n+2}^y = 3Q_{n+1}^y - Q_n^y$ *with* $Q_0^y = 0,\ Q_1^y = 1,\ Q_2^y = 3,\ \ldots$
Therefore, $Q_{n+1}^y = P_{n+1}^x = S_n(3)$.

(II) $P_{n+2}^y = 3P_{n+1}^y - P_n^y + 1$ *with* $P_0^y = 0,\ P_1^y = 0,\ P_2^y = 1,\ \ldots$
Similarly, $Q_{n+2}^x = 3Q_{n+1}^x - Q_n^x + 1$ *with* $Q_0^x = 0,\ Q_1^x = 0,\ Q_2^x = 1,\ \ldots$
Therefore we have $P_{n+1}^y = Q_n^x = P_{n+1}^x - P_n^x - 1 = S_n(3) - S_{n-1}(3) - 1$.

Let F_k be the Fibonacci sequence defined by $F_0 = 0,\ F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$. It relates to Chebyshev polynomial as follows:

Lemma 2.5. *We have* $F_{2n} = S_{n-1}(3)$.

Proof. $F_{2n} = F_{2n-1} + F_{2n-2} = 2F_{2n-2} + F_{2n-3} = 3F_{2n-2} - F_{2n-4}$. Thus F_{2n} and $S_{n-1}(3)$ satisfy the same recursive relation. Noting that $F_2 = 1 = S_0(3)$ and $F_4 = 3 = S_1(3)$ we conclude the lemma. □

The main result of this section can be expressed using Fibonacci number as follows:

Theorem 2.6. *Let D_n be the closure of the braid $(\sigma_1\sigma_2^{-1})^n$*
Then:

$$Col^{red}(D_n) = \begin{cases} \mathbb{Z}_{F_{n-1}+F_{n+1}} \oplus \mathbb{Z}_{F_{n-1}+F_{n+1}} & \textit{when } n \textit{ is odd,} \\ \mathbb{Z}_{5F_n} \oplus \mathbb{Z}_{F_n} & \textit{when } n \textit{ is even} \end{cases}$$

The sums $F_{n-1} + F_{n+1}$ are called Lucas numbers denoted by L_n. That is $L_0 = 2, L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, ...$ and $L_{n+2} = L_{n+1} + L_n$.

We divide the proof of Theorem 2.6 into three lemmas.

Lemma 2.7. *The matrix of relations for $Col^{red}(D_n)$ is*

$$A_{2n} = \begin{bmatrix} F_{2n} & F_{2n-1} - 1 \\ F_{2n+1} - 1 & F_{2n} \end{bmatrix}$$

Proof. From Lemma 2.3, we know $Col^{red}(D_n)= \begin{bmatrix} P_n^x & P_n^y \\ Q_n^x & Q_n^y \end{bmatrix} = \begin{bmatrix} P_n^x & P_n^x - P_{n-1}^x - 1 \\ P_{n+1}^x - P_n^x - 1 & P_n^x \end{bmatrix}$.

Now by Lemma 2.4, $P_n^x - P_{n-1}^x - 1 = S_{n-1}(3) - S_{n-2}(3) - 1 = F_{2n} - F_{2n-2} - 1 = F_{2n-1} - 1$. Therefore $P_{n+1}^x - P_n^x - 1 = F_{2n+2} - F_{2n} - 1 = F_{2n+1} - 1$.

Hence, $Col^{red}(D_n) = \begin{bmatrix} F_{2n} & F_{2n-1} - 1 \\ F_{2n+1} - 1 & F_{2n} \end{bmatrix}$. We denote this matrix A_{2n}.

□

Lemma 2.8. $Col^{red}(D_n) = \mathbb{Z}_{F_{n-1}+F_{n+1}} \oplus \mathbb{Z}_{F_{n-1}+F_{n+1}}$ *when n is odd.*

Proof. (1) We replace the first column in A_{2n} by the difference of the first and the second column to get:

$$A_{2n-1} = \begin{bmatrix} F_{2n-2} + 1 & F_{2n-1} - 1 \\ F_{2n-1} - 1 & F_{2n} \end{bmatrix}$$

(2) We replace the second column in A_{2n-1} by the difference of the second column and the first column to get:

$$A_{2n-2} = \begin{bmatrix} F_{2n-2} + 1 & F_{2n-3} - 2 \\ F_{2n-1} - 1 & F_{2n-2} + 1 \end{bmatrix}$$

We generalize (1) and (2) to arbitrary number of column operations (with the same pattern of operations):

After performing k operations of (1) and (2), we get

$$A_{2n-2k} = \begin{bmatrix} F_{2n-2k} + F_{2k} & F_{2n-(2k+1)} - F_{2k+1} \\ F_{2n-(2k-1)} - F_{(2k-1)} & F_{2n-2k} + F_{2k} \end{bmatrix}$$

Since n is odd, let $k = \frac{n+1}{2}$; then we have:

$$A_{n-1} = \begin{bmatrix} F_{n-1} + F_{n+1} & F_{n-2} - F_{n+2} \\ 0 & F_{n-1} + F_{n+1} \end{bmatrix}$$

Hence, the gcd of $Col^{red}(D)$ (that is, the gcd of entries of the matix) is $F_{n-1} + F_{n+1}$ and $detA_{n-1} = (F_{n-1} + F_{n+1})^2$. Therefore $Col^{red}(D_n) = \mathbb{Z}_{F_{n-1}+F_{n+1}} \oplus \mathbb{Z}_{F_{n-1}+F_{n+1}}$ for n odd. [3]

[3] We use the standard fact that for $\mathbb{Z}$ modules given by 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The related abelian group is equal to $\mathbb{Z}_{d/g} \oplus \mathbb{Z}_g$ where d is the determinant of the matrix and g is $gcd(a, b, c, d)$.

∎

Lemma 2.9. *$Col^{red}(D_n) = \mathbb{Z}_{5F_n} \oplus \mathbb{Z}_{F_n}$ when n is even.*

Proof. Let $k = \frac{n}{2}$ and use the proof of Lemma 2.8.

Then we have: $A_n = \begin{bmatrix} 2F_n & F_{n-1} - F_{n+1} \\ F_{n+1} - F_{n-1} & 2F_n \end{bmatrix} = \begin{bmatrix} 2F_n & -F_n \\ F_n & 2F_n \end{bmatrix}$

Adding twice the last column of the A_n to the first, we get: $\begin{bmatrix} 0 & -F_n \\ 5F_n & 2F_n \end{bmatrix}$.

Here the gcd of $Col^{red}(D)$ is F_n and $|Col^{red}(D)| = 5F_n^2$. Thus we have $Col^{red}(D_n) = \mathbb{Z}_{5F_n} \oplus \mathbb{Z}_{F_n}$ for n even.

∎

Remark 2.10. *In* [Prz2] *the Goeritz matrix of $D(W_n)$ was reduced to*

$$\begin{bmatrix} S_{n-1}(3) & 1 - S_n(3) \\ S_{n-2}(3) + 1 & -S_{n-1}(3) \end{bmatrix}$$

To show that the group it presents is the same as before, we multiply the last column by -1 and use the equality $S_{n-1}(3) = F_{2n}$, to get:

$$\begin{bmatrix} F_{2n} & F_{2n+2} - 1 \\ F_{2n-2} + 1 & F_{2n} \end{bmatrix} = \begin{bmatrix} F_{2n} & F_{2n+1} + F_{2n} - 1 \\ F_{2n-2} + 1 & F_{2n-1} + F_{2n-2} \end{bmatrix}$$

After a column operation we get the matrix

$$\begin{bmatrix} F_{2n} & F_{2n+1} - 1 \\ F_{2n-2} + 1 & F_{2n-1} - 1 \end{bmatrix} = \begin{bmatrix} F_{2n-1} + F_{2n-2} & F_{2n} + F_{2n-1} - 1 \\ F_{2n-2} + 1 & F_{2n-1} - 1 \end{bmatrix}$$

After a row operation we get the matrix

$$\begin{bmatrix} F_{2n-1} - 1 & F_{2n} - 1 \\ F_{2n-2} + 1 & F_{2n-1} - 1 \end{bmatrix}$$

which is the matrix considered in (1) above with row exchanged so leading to the same abelian group as A_{2n}

Remark 2.11. *We can deduce several formulas from our column reductions. For example*

$$F_{2n} = F_n(F_{n-1} + F_{n+1}).$$

and

$$F_{2n-1} - 1 = \begin{cases} F_n(F_{n-2} + F_n) & \text{for } n \text{ even} \\ F_{n-1}(F_{n-1} + F_{n+1}) & \text{for } n \text{ odd} \end{cases}$$

The identities in Corollary 2.11 are well known; e.g. they follow from identities in [Koshy] page 97. In particular, $F_{2n} = F_n(F_{n-1} + F_{n+1})$ is exactly identity 54 of [Koshy].[4] Thomas Koshy mentions that the identities follow directly from closed forms for F_n and L_n. We present the proof for completeness. We consider roots of the polynomial $x^2 - x - 1 = 0$ say α and β, and note that

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n.$$

We have $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = -\alpha^{-1} = \frac{1-\sqrt{5}}{2}$, $\alpha - \beta = \sqrt{5}$. Our identities can be expressed as product to sum formulas (using $F_{-k} = (-1)^{k+1}F_k$):

Proposition 2.12.

$$F_m(F_{n-1} + F_{n+1}) = F_m L_n = F_{m+n} + (-1)^n F_{m-n} = F_{m+n} - (-1)^m F_{n-m}.$$

Proof.

$$\begin{aligned} F_m L_n &= \frac{(\alpha^m - \beta^m)(\alpha^n + \beta^n)}{\alpha - \beta} \\ &= \frac{\alpha^{m+n} - \beta^{m+n} + \alpha^m\beta^n - \alpha^n\beta^n}{\alpha - \beta} \\ &= \frac{\alpha^{m+n} - \beta^{m+n} + (-1)^n(\alpha^{m-n} - \beta^{m-n})}{\alpha - \beta} \\ &= F_{m+n} + (-1)^n F_{m-n}. \end{aligned}$$

□

[4] There are also the following related identities:

Identity 52: $F_{2m+n} - (-1)^m F_n = F_m L_{m+n}$,

Identity 53: $F_{2m+n} + (-1)^m F_n = F_{m+n} L_m$,

3. Alexander-Burau-Fox matrix of a link diagram

In this chapter we generalize our main result of the second section from the Fox group of colorings to the Alexander-Burau-Fox module over the ring $\mathbb{Z}[t^{\pm 1}]$. Where the Fibonacci numbers appeared in the computation of the the Fox Coloring Group, the analogous combinatorial object appearing in the computation of the ABF Module is the Chebyshev Polynomial of the second kind. The modules we study are Alexander modules of links [Ale] in a form which can be deduced from the Burau representation of braids [Burau, Bur-Z]. These module directly generalize to the group of Fox colorings where we replace -1 by t.

The formal definition of the ABF Module follows (notice that for $t = -1$, we get Definition 1.4).

Definition 3.1. *Let D be an oriented link diagram. The $\mathbb{Z}[t^{\pm 1}]$-module $ABF(D)$ has a presentation where generators are indexed by the arcs of D, denoted by $Arcs(D)$, and whose relations are given by crossings of D (positive and negative) as follows*

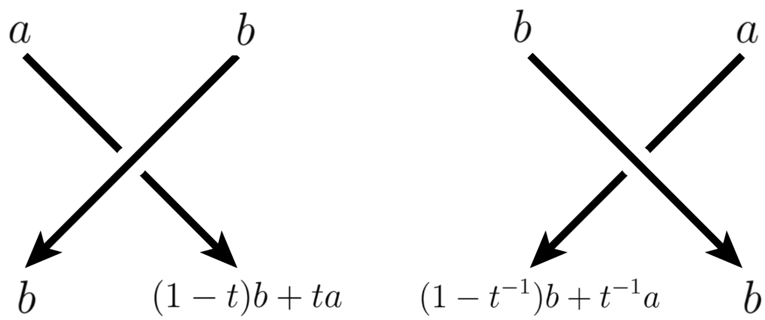
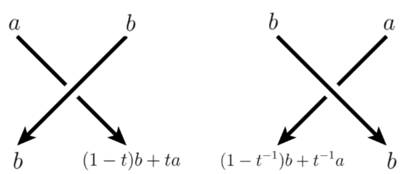
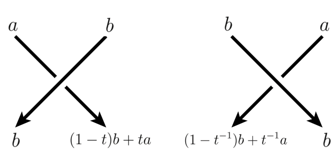


Figure 5. ABF Module Relations

That is,

$ABF_n(D) = \{Arcs(D) \mid$  *where* $a, b, c \in Arcs(D)\}$

Definition 3.2. *The Reduced Alexander-Burau-Fox Module is the module*

$ABF_n^{Red}(D) = \{Arcs(D) \mid$  $, \sum_{a_i \in arcs(D)} a_i = 0\},$

where the sum is taken over all arcs of D. Note that this module does eliminates trivial colorings, which is equivalent to setting one arc, say a_s equal to zero (removing one generator and adjusting relations by substituting $a_s = -\sum_{a_i, i \neq s} a_i$).[5]

[5]To be precise: the relation $\sum_{a_i \in arcs(D)} a_i = 0$ allows us to replace any other relation so that the coefficient of a_s is equal to zero. In this new presentation a_s can be removed and the presentation has generators $Arcs(D) - a_s$, and in relations we can put $a_s = 0$.

We now formulate the main result of this section, showing the structure of the ABF module of the diagrams of wheel graphs. We simplify our notation by calling $ABF(D(W_n)) = M_n(D_n)$. Notice that the module decomposes as the sum of cyclic modules (even though $\mathbb{Z}[t^{\pm 1}]$ is not a PID), and that for odd n the module is double and for even n it is "almost" double.

Theorem 3.3. *The reduced Alexander-Burau-Fox module of diagrams of wheel graphs W_n can be expressed as*

$$M_n^{Red}(D(W_n)) = \begin{cases} \mathbb{Z}[t^{\pm 1}]/(S_{k-1}+S_k) \oplus \mathbb{Z}[t^{\pm 1}]/(S_{k-1}+S_k), & \text{when } n = 2k+1 \\ \mathbb{Z}[t^{\pm 1}]/(S_{k-1}) \oplus \mathbb{Z}[t^{\pm 1}]/((3-t-t^{-1})S_{k-1}) & \text{when } n = 2k \end{cases}$$

where $z = 1-t-t^{-1}$ and $S_k(z)$ is the Chebyshev polynomial of the second kind (see Subsection 3.1)

As before, we label the inputs of our braids a, b, c and the outputs a', b', c'; see Figure 6.

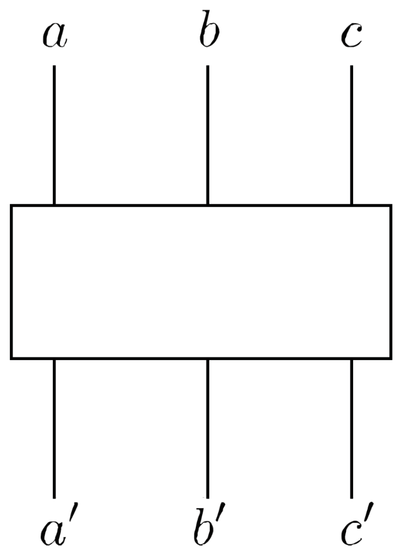


Figure 6. Coloring of the tangle

As a special case of the general observation for n-tangles, we have that $t^2(a'-a)+t(b'-b)+(c'-c)=0$. Thus $b'-b = -t(a'-a)-t^{-1}(c'-c)$, meaning that the middle color can be computed from the first and the third.

Now like in the $t=-1$ case, let the endpoints of $(\sigma_1\sigma_2^{-1})^n$ be $a' = P_n + a$ on the left and $c' = Q_n + c$ on the right. Then $b'-b = -tP_n - t^{-1}Q_n$. As mentioned before, comoputing the Reduced ABF module is equivalent to setting $b=0$. Thus, we have that $b' = -tP_n - t^{-1}Q_n$.

Writing $P_n = P_n^a \cdot a + P_n^c \cdot c$ and $Q_n = Q_n^a \cdot a + Q_n^c \cdot c$ as before, we see that the matrix of relations for the Reduced Alexander-Burau-Fox module, denoted A_n, is analogous to that of the Reduced Fox Coloring Group:

$$A_n = \begin{bmatrix} P_n^a & P_n^c \\ Q_n^a & Q_n^c \end{bmatrix}.$$

We also produce the following recursive relations, analogous to those found in Section 2. That is,

Lemma 3.4. (1) $P_{n+1} = -tP_n - t^{-1}Q_n - a$,
(2) $Q_{n+1} = (1-t)P_{n+1} + tP_n + a - c$.

Proof. Considering we work with the reduced ABF Module, we set $b = 0$. Let b'_n denote the central strand after performing $(\sigma_1\sigma_2^{-1})$ n times. By previous observation, $b'_n = -tP_n - t^{-1}Q_n$. Then we have $P_{n+1} + a = b'_n$, so $P_{n+1} + a = b' = -tP_n - t^{-1}Q_n$. This proves (1) as is shown in Figure 7.

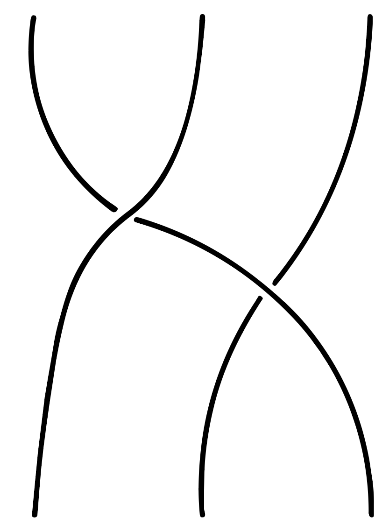


FIGURE 7. Labeling part of $(\sigma_1\sigma_2^{-1})^n$ in ABF Module

Next, we see that

$$\begin{aligned} Q_{n+1} + c &= (1+t)(-tP_n - t^{-1}Q_n) + t(P_n + a) \\ &= -tP_n - t^{-1}Q_n - t^2P_n - Q_n + tP_n + a \\ &= P_{n+1} - t(tP_n - t^{-1}Q_n) + tP_n + a \\ &= P_{n+1} - tP_{n+1} + tP_n + a \\ \implies Q_{n+1} &= (1-t)P_{n+1} + tP_{n+a}. \end{aligned}$$

Thus, (2) is true. □

Immediately from the Lemma 3.8, we can present the matrix of relations A_n in terms of P_n^a and P_n^c by the following lemma.

Lemma 3.5. *For Q_n^a and Q_n^c as defined above,*

(1) $Q_n^a = (1-t)P_n^a + tP_{n-1}^a + 1$
(2) $Q_n^c = (1-t)P_n^c + tP_{n-1}^c - 1$

Proof. By Lemma 3.4 (2),

$$\begin{aligned}
Q_n &= Q_n^a \cdot a + Q_n^c \cdot c \\
&= (1-t)P_{n+1} + tP_n + a - c \\
&= (1-t)P_n^a \cdot a + tP_{n-1}^a \cdot a + a + (1-t)P_n^c \cdot c + tP_{n-1}^c \cdot c - c \\
&= ((1-t)P_n^a + tP_{n-1}^a + 1) \cdot a + ((1-t)P_n^c + tP_{n-1}^c - 1) \cdot c.
\end{aligned}$$

□

Thus, our Alexander-Fox-Burau matrix is

$$A_n = \begin{bmatrix} P_n^a & P_n^c \\ Q_n^a & Q_n^c \end{bmatrix} = \begin{bmatrix} P_n^a & P_n^c \\ (1-t)P_n^a + tP_{n-1}^a + 1 & (1-t)P_n^c + tP_{n-1}^c - 1 \end{bmatrix}.$$

where $z = 1 - t - t^{-1}$.

The final identity we prove in service of our eventual translation of this matrix into the language of Chebyshev polynomials is the following:

Lemma 3.6. *For P_n^a and P_n^c we have the following Chebyshev-type recursive relation:*

(1) $P_n^a = zP_{n-1}^a - P_{n-2}^a - (1 + t^{-1})$
(2) $P_n^c = zP_{n-1} - P_{n-2}^c + t^{-1}$

Proof. Beginning with Lemma 3.8 (1),

$$\begin{aligned}
P_n &= P_n^a \cdot a + P_n^c \cdot c \\
&= -tP_{n-1} - t^{-1}Q_{n-1} - a \\
&= -tP_{n-1} - t^{-1}((1-t)P_{n-1} + tP_{n-2} + a - c) - a \\
&= -tP_{n-1} - t^{-1}P_{n-1} + P_{n-1} - P_{n-2} - t^{-1}a + t^{-1}c - a \\
&= (1 - t - t^{-1})P_{n-1} - P_{n-2} - t^{-1}a + t^{-1}c - a \\
&= (zP_{n-1}^a - P_{n-2}^a - t^{-1} - 1) \cdot a - (zP_{n-1}^c - P_{n-2}^c + t^{-1}) \cdot c
\end{aligned}$$

□

3.1. **Chebyshev Polynomials.** We now set about rewriting P_n^a and P_n^c in terms of Chebyshev polynomials using Theorem 3.8, below.

We recall first the definition of Chebyshev Polynomials of the Second Kind.

Definition 3.7. *Let*

$$\begin{aligned}
&S_0(z) = 1, \\
&S_1(z) = z, \text{ and} \\
&S_n(z) = zS_{n-1}(z) - S_{n-2}(z).
\end{aligned}$$

Then $S_n(z)$ is the n-th polynomial of the Second Kind.

Throughout the paper, we evaluate $S_n(z)$ at $z = 1 - t - t^{-1}$, though we tend to suppress the notation: $S_n(z) \to S_n$.

Theorem 3.8. *Let the sequence of polynomials $P_n(z, c_i) \in \mathbb{Z}[z, c_i]$ satisfy the recursive relation $P_n = zP_{n-1} - P_{n-2} + c_i$. Then P_n can be expressed using Chebyshev polynomials $S_n(z)$ as follows:*

$$P_n = S_{n-1}P_1 - S_{n-2}P_0 + \sum_{j=0}^{n-2} S_j c_{n-j}.$$

Proof. We proceed by induction on n. Let $n = 3$ be our base case. Then

$$\begin{aligned} P_3 &= zP_2 - P_1 + c_3 \\ &= z(zP_1 - P_0 + c_2) - P_1 + c_3 \\ &= z^2P_1 - zP_0 + zc_2 - P_1 + c_3 \\ &= (z^2 - 1)P_1 - zP_0 + zc_2 + c_3 \\ &= S_2P_1 - S_1P_0 + S_0c_3 + S_1c_2 \\ &= S_2P_1 - S_1P_0 + \sum_{j=0}^{1} S_j c_{n-j}. \end{aligned}$$

Assume that the proposition holds for $n \geq 3$ up to $n - 1$.

By inductive hypothesis,

$$\begin{aligned} P_n &= zP_{n-1} - P_{n-2} + c_n \\ &= z(S_{n-2}P_1 - S_{n-3}P_0 + \sum_{j=1}^{n-3} S_j c_{n-1-j}) - (S_{n-3}P_1 - S_{n-4}P_0 + \sum_{j=0}^{n-4} S_j c_{n-2-j}) + c_n \\ &= (zS_{n-2} - S_{n-3})P_1 - (zS_{n-3} - S_{n-4})P_0 + z\sum_{j=0}^{n-3} S_j c_{n-1-j} - \sum_{j=0}^{n-4} S_j c_{n-2-j} + c_n \\ &= S_{n-1}P - S_{n-2}P_0 + \sum_{j=0}^{n-3} S_j c_{n-1-j} - \sum_{j=0}^{n-4} S_j c_{n-2-j} + c_n \\ &= S_{n-1}P - S_{n-2}P_0 + zS_0c_{n-1} + \sum_{j=2}^{n-2} S_j c_{n-j} + c_n \\ &= S_{n-1}P - S_{n-2}P_0 + S_0c_n + S_1c_{n-1} + \sum_{j=2}^{n-2} S_j c_{n-j} \\ &= S_{n-1}P - S_{n-2}P_0 + \sum_{j=0}^{n-2} S_j c_{n-j}. \end{aligned}$$

Thus, our identity holds.

∎

Thus we can rewrite the recursive relations in Lemma 3.6 with the Corollary 3.9.

Corollary 3.9. (1) $P_n^a(t) = -S_{n-1} - (1 + t^{-1}) \sum_{j=0}^{n-2} S_j(z)$.

(2) $P_n^c(t) = t^{-1} \sum_{j=0}^{n-2} S_j(z)$ *for* $z = 1 - t - t^{-1}$.

Proof. (1) By Lemma 3.6 (1), $P_n^a = zP_{n-1}^a - P_{n-2}^a - (1+t^{-1})$, so applying Theorem 3.8 with $c = -1 - t^{-1}$ yields the result with initial conditions $P_0^a = 0$ and $P_1^a = -1$.
(2) By 3.6 (2), $P_n^c = zP_{n-1}^c - P_{n-2}^c + t^{-1}$ with initial conditions $P_0^c = 0$ and $P_1^c = t^{-1}$. □

In the next section, we simply the equation in Corollary 3.9 using Product to Sum formulas (in fact, we go from sum to product).

3.2. **Product to Sum Formulas.** Seeking further transformation of A_n, we provide the following identities for Chebyshev polynomials, analogous to those given for the Fibonacci numbers.

Here we let S_n be any Chebyshev polynomial of the second kind, and T_n a Chebyshev polynomial of the first kind. That is, $T_0 = 1, T_1 = x$ and $T_n = xT_{n-1} - T_n$. After substituting $x = p + p^{-1}$ we obtain

$$S_n = \frac{p^{n+1} - p^{-n-1}}{p - p^{-1}}$$

and

$$T_n = p^n + p^{-n}.$$

These formulas will be used to prove the following well-known product to sum formulas. We offer the proof for completeness.

Proposition 3.10. *For S_n and T_n as defined above,*

(1) $T_mT_n = T_{m+n} + T_{m-n}$
(2) $S_mT_n = S_{m+n} + S_{m-n}$
(3) $S_mS_n = S_{m+n} + S_{m+n-2} + ... + S_{m-n+2} + S_{m-n} = \sum_{i=m-n, i\equiv m+n \mod 2}^{m+n} S_i$ *where the sum is taken over i with $i \equiv m+n \mod 2$.*
(4) $S_n(S_n + S_{n-1}) = S_0 + S_1 + ... + S_{2n-1} + S_{2n} = \sum_{i=0}^{2n} S_i$.
(5) $S_n(S_n + S_{n+1}) = S_0 + S_1 + ... + S_{2n-1} + S_{2n} + S_{2n+1} = \sum_{i=0}^{2n+1} S_i$.

Proof. (1): $T_mT_n = (p^m + p^{-m})(p^n + p^{-n}) = p^{m+n} + p^{-m-n} + p^{m-n} + p^{n-m} = T_{m+n} + T_{m-n}$.

(2): $S_mT_n = \frac{(p^{m+1} - p^{-m-1})(p^n + p^{-n})}{p - p^{-1}} = \frac{(p^{m+n+1} - p^{-m-n-1}) + (p^{m-n+1} - p^{-m+n-1}}{p - p^{-1})} = T_{m+n} + T_{m-n}$.

(3): We proceed by induction on n, applying (2) repeatedly:
If $n = 0$ we have $S_mS_0 = S_m$ and for $m = 1$, $S_mS_1 = xS_m = S_{m+1} + S_{m-1}$. Inductive step holds as follows: (assuming $m \geq n \geq 2$): $S_mS_n = S_m(S_n - S_{n-2}) + S_mS_{n-2} = S_mT_n + S_mS_{n-2} \overset{(2)}{=} S_{m+n} + S_{m-n} + S_mS_{n-2} \overset{ind}{=} S_{m+n} + S_{m-n} + S_{m+n-2} + S_{m+n-4} + ... + S_{m-n+2}$, as needed.
(4) We use (3) twice for S_nS_n and for S_nS_{n-1}.
(5) We use (3) twice for S_nS_n and for S_nS_{n+1}.

□

The following two corollaries further simplify A_n.

Corollary 3.11.

$$\sum_{j=0}^{n-2} S_j = \begin{cases} S_k(S_k + S_{k-1}) & \textit{when } n = 2k, \\ S_{k-1}(S_{k-1} + S_k) & \textit{when } n = 2k+1. \end{cases}$$

Proof. This is a restatement of (4) and (5) from Proposition 3.10. □

Corollary 3.12. (1) $S_{2k} = S_k S_k - S_{k-1}S_{k-1} = (S_k - S_{k-1})(S_k + S_{k-1})$
(2) $S_{2k+1} = S_k S_{k+1} - S_{k-1}S_k = S_k(S_{k+1} - S_{k-1})$.

Proof. (1) By 3.10 (3),

$$\begin{aligned} S_k S_k &= S_0 + S_2 + ... + S_{2k} \\ &= (S_0 + S_2 + \cdots + S_{2k-2}) + S_{2k} \\ &= S_{k-1}S_{k-1} + S_{2k} \end{aligned}$$

So $S_k S_k = S_{k-1}S_{k-1} + S_{2k}$ implies $S_{2k} = S_k S_k - S_{k-1}S_{k-1}$.

(2)

$$S_k S_{k+1} = S_1 + S_3 + ... + S_{2k+1},$$

□

This allows us to reformulate P_n^a and P_n^c.

Lemma 3.13.

(1) $P_{2k}^a = -S_{k-1}((S_k + S_{k-1}) + t^{-1}(S_{k-1} + S_{k-2}))$
(2) $P_{2k+1}^a = -(S_{k-1} + S_k)(S_k + t^{-1}S_{k-1})$
(3) $P_{2k}^c = t^{-1}S_{k-1}(S_{k-1} + S_{k-2})$
(4) $P_{2k+1}^c = t^{-1}S_{k-1}(S_{k-1} + S_k)$

Proof.

(1) By Corollary 3.9, $P_{2k}^a(t) = -S_{2k-1}(z) - (1 + t^{-1})\sum_{j=0}^{2k-2} S_j(z)$, where $z = 1 - t - t^{-1}$, as before. Thus,

$$\begin{aligned} P_{2k}^a &= -S_{2k-1} - (1 + t^{-1}) \sum_{j=0}^{2(k-1)} S_j(z) \\ &= -S_{2k-1} - (1 + t^{-1})S_{k-1}(S_{k-1} + S_{k-2}) && \text{Cor. 3.11} \\ &= -S_{2(k-1)+1} - (1 + t^{-1})S_{k-1}(S_{k-1} + S_{k-2}) \\ &= -S_{k-1}(S_k - S_{k-2}) - (1 + t^{-1})S_{k-1}(S_{k-1} + S_{k-2}) && \text{Cor. 3.12} \\ &= -S_{k-1}(S_k - S_{k-2} + S_{k-1} + S_{k-2} + t^{-1}S_{k-1} + t^{-1}S_{k-2}) \\ &= -S_{k-1}((S_k + S_{k-1}) - t^{-1}(S_{k-1} + S_{k-2})) \end{aligned}$$

(2) By Corollary 3.9, $P^a_{2k}(t)$

$$
\begin{aligned}
-P^a_{2k+1} &= -S_{2k+1-1} - (1+t^{-1}) \sum_{j=0}^{2k+1-2} S_j(z) \\
&= -S_{2k} - (1+t^{-1})S_{k-1}(S_{k-1}+S_{k+1}) && \text{Cor. 3.11} \\
&= -(S_k - S_{k-1})(S_k + S_{k-1}) - (1+t^{-1})S_{k-1}(S_{k-1}+S_k) && \text{Cor. 3.12} \\
&= -(S_k + S_{k-1})(S_k - S_{k-1} + S_{k-1} + t^{-1}S_{k-1}) \\
&= -(S_k + S_{k-1})(S_k + t^{-1}S_{k-1})
\end{aligned}
$$

(3) & (4) These follow immediately from Cor 3.11.

□

Now, we present the Alexander-Burau-Fox matrix for closures of the braids of type $(\sigma_1\sigma_2^{-1})^n$ in full generality.

Proposition 3.14.

$$
A_n = \begin{bmatrix} P^a_n & P^c_n \\ Q^a_n & Q^c_n \end{bmatrix} = \begin{bmatrix} -g_n(g_{n+1} - t^{-1}g_{n-1}) & t^{-1}g_n g_{n-1} \\ t g_n g_{n+1} & -g_n(g_{n+1} - t g_{n-1}) \end{bmatrix}
$$

where

$$
g_n = \begin{cases} S_{k-1} & \text{when } n = 2k, \\ S_{k-1} + S_k & \text{when } n = 2k+1. \end{cases}
$$

Proof. This follows immediately from Lemma 3.13 and Lemma 3.5. □

3.3. **Alexander-Burau-Fox Module.** With the explicit presentation of the matrix A_n in terms of Chebyshev polynomials from 3.14, we compute the resulting module by row reduction.

Define

$$
A'_n = A_n/(-g_n) = \begin{bmatrix} g_{n+1} - t^{-1}g_{n-1} & -t^{-1}g_{n-1} \\ -t g_{n+1} & g_{n+1} - t g_{n-1} \end{bmatrix}.
$$

One column operation, $\mathrm{col}_1 - \mathrm{col}_2 \to \mathrm{col}_1$, yields

$$
A'_n = \begin{bmatrix} g_{n+1} & -t^{-1}g_{n-1} \\ (-t-1)g_{n+1} + t g_{n-1} & g_{n+1} - t g_{n-1} \end{bmatrix}
$$

We multiply the second column by $-t$:

$$
A'_n = \begin{bmatrix} g_{n+1} & g_{n-1} \\ (-t-1)g_{n+1} + t g_{n-1} & -t g_{n+1} + t^2 g_{n-1} \end{bmatrix}
$$

Specifically for $n = 2k+1$, we get

$$
A'_{2k+1} = \begin{bmatrix} S_k & S_{k-1} \\ -tS_k & -tS_k + t^2 S_{k-1}] \end{bmatrix}.
$$

For $n = 2k$, we get

$$
A'_{2k} = \begin{bmatrix} S_{k-1} + S_k & S_{k-1} + S_{k-2} \\ -t(S_{k-1} + S_k) & -t(S_{k-1} + S_k) - t^2(S_{k-1} + S_{k-2}) \end{bmatrix}.
$$

We will apply the following lemma to A''_n in order to immediately compute the desired module.

Lemma 3.15. (1) *The pair* (S_k, S_{k-1}) *can be reduced by Euclidean algorithm (column operations) to* $(S_0, S_{-1}) = (1, 0)$.

(2) *The pair* $(S_k + S_{k-1}, S_{k-1} + S_{k-2})$ *can be reduced by Euclidean algorithm (column operations) to* $(S_1 + S_0, S_0 + S_{-1}) = (z + 1, 1)$.

Proof. (1) We notice that $(S_k, S_{k-1}) = (zS_{k-1} - S_{k-2}, S_{k-1})$ thus by taking the first column minus z times the second we get $(-S_{k-2}, S_{k-1})$. Multiplying the first column by -1 we get (S_{k-2}, S_{k-1}) allowing inductive step.
(2) We notice that $(S_k + S_{k-1}, S_{k-1} + S_{k-2}) = (zS_{k-1} - S_{k-2} + zS_{k-2} - S_{k-3}, S_{k-1} + S_{k-2}) = (z(S_{k-1} + S_{k-2}) - (S_{k-2} + S_{k-3}), S_{k-1} + S_{k-2})$, thus by taking the first column minus z times the second we get $(-(S_{k-2} + S_{k-3}), S_{k-1} + S_{k-2})$ and multiplying the first column by -1 we get $(S_{k-2} + S_{k-3}, S_{k-1} + S_{k-2})$, allowing the inductive step. ☐

We apply Lemma 3.15 to A'_n. After applying this lemma we get the matrix

$$\begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \text{ reduced by row operation to } \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}$$

where $y = detA'_n$.
Thus we can conclude that our Alexander-Burau-Fox module given by A_n is:

$$\mathbb{Z}[t^{\pm 1}]/(g_n) \oplus \mathbb{Z}[t^{\pm 1}]/(detA'_n \cdot g_n).$$

The $\det(A'_n)$ is computed in the next section.

3.4. Computing determinant of the matrix A'_n.

With this computation the main result on the structure of the Alexander-Burau-Fox module of $(\sigma_1\sigma_2^{-1})^n$ is complete.

Proposition 3.16. *For* A'

(1) $detA'_{2k+1} = 1$.
(2) $detA'_{2k} = 3 - t - t^{-1}$.

Proof. (1) We have

$$\begin{aligned} detA'_{2k+1} &= (S_k + t^{-1}S_{k-1})(S_k + tS_{k-1}) - S_kS_{k-1} \\ &= S_k^2 + (t + t^{-1})S_kS_{k-1} + S_{k-1}^2 - S_kS_{k-1} \\ &= S_k^2 + (t + t^{-1} - 1)S_kS_{k-1} + S_{k-1}^2 = S_k^2 - zS_kS_{k-1} + S_{k-1}^2 \\ &= S_k^2 - (S_{k+1} + S_{k-1})S_{k-1} + S_{k-1}^2 \\ &= S_k^2 - S_{k-1}S_{k+1} \\ &= 1 \end{aligned}$$

by product to sum formulas. (2) We proceed as in the previous case. We have

$$
\begin{aligned}
det A'_{2k} &= \Big((S_k + S_{k-1}) + t^{-1}(S_{k-1} + S_{k-2})\Big)\Big((S_k + S_{k-1})) + t(S_{k-1} + S_{k-2})\Big) \\
&\quad - (S_k + S_{k-1})(S_{k-1} + S_{k-2}) \\
&= (S_k + S_{k-1})^2 + (t + t^{-1})(S_k + S_{k-1})(S_{k-1} + S_{k-2}) + (S_{k-1} + S_{k-2})^2 - (S_k + S_{k-1})(S_{k-1} + S_{k-2}) \\
&= (S_k + S_{k-1})^2 + (t + t^{-1} - 1)(S_k + S_{k-2})(S_{k-1} + S_{k-2}) + (S_{k-1} + S_{k-2})^2 \\
&= (S_k + S_{k-1})^2 - z(S_k + S_{k-1})(S_{k-1} + S_{k-2}) + (S_{k-1} + S_{k-2})^2 \\
&= (S_k + S_{k-1})^2 - ((S_{k+1} + S_{k-1}) + (S_k + S_{k-2}))(S_{k-1} + S_{k-2}) + (S_{k-1} + S_{k-2})^2 \\
&= (S_k + S_{k-1})^2 - (S_{k+1} + S_k)(S_{k-1} + S_{k-2}) \\
&= (S_k S_{k-1} - S_{k+1} S_{k-3}) + (S_k S_k - S_{k+1} S_{k-1}) + (S_{k-1} S_{k-1} - S_k S_{k-2}) \\
&= S_1 + S_0 + S_0 \\
&= z + 2 \\
&= 1 - t - t^{-1} + 2 \\
&= 3 - t - t^{-1}.
\end{aligned}
$$

□

Thus, with the computation of $\det(A'_n)$, we complete the proof of the main theorem, Theorem 3.3.

4. Relation to Burau representation

The crossing illustrated in Figure 5 can be interpreted as a 2-braid, and the ABF relation as a linear map: for a positive crossing, $B(a,b) = (b, ta + (1-t)b$. This map is given in a standard basis $(1,0),(0,1)$ by the matrix:

$$B = \begin{bmatrix} 0 & 1 \\ t & 1-t \end{bmatrix}$$

which is an element of $GL(2, Z[t\pm 1])$. More generally, for $n-$ braid we have a homomorphism $B_n \to GL(n, Z[t \pm 1])$ given on generators by

$$B(\sigma_i) = \begin{bmatrix} Id_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t & 1-t & 0 \\ 0 & 0 & 0 & Id_{n-i-1} \end{bmatrix}$$

This representation is know as the (unreduced) Burau representation [Burau]. Because for the negative crossing we have $B(b,a) = ((1-t^{-1})b, t^{-1}a)$ so

$$B(\sigma_i^{-1}) = \begin{bmatrix} Id_{i-1} & 0 & 0 & 0 \\ 0 & 1-t^{-1} & t^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & Id_{n-i-1} \end{bmatrix}$$

Clearly $B(\sigma_i)B(\sigma_i^{-1}) = Id_n$ reflecting second Reidemeister move. In older literature the role of σ_i and σ_i^{-1} (and t and t^{-1}) is changed (historically negative and positive was not always the same and till know braid theorists use the opposite convention from knot theorists).

The matrix $B(\gamma) - Id_n$ where $\gamma \in B_n$ gives (with relations in rows) description of the unreduced Alexander-Burau-Fox module. This is, in fact, the matrix used in our calculation in Chapter 3.

5. Speculations and Future Directions

To place our result in a broader context, we should mention the classical result of J.Minkus and Mulazzani-Vesnin [Min, MV] relating n-fold branch coverings of 2-bridge links with double branch coverings of certain links. In our case, he notices that the n-fold branch coverings of S^3 along the figure eight knot is homeomorphic to the double branch cover of S^3 branched along the closure of the braid $(\sigma_1\sigma_2^{-1})^n$. Thus our theorem on the group of Fox colorings can be reformulated in the language of homology of $M_{4_1}^{(n)}$, where $M_L^{(n)}$ denotes the n-fold branch coverings of S^3 branched along the link L. We plan to explore this connection to analyze left orderings of the fundamental groups of some branched coverings; compare [DaPr]. We recall the results of [BGW], which show that the 2-fold branch coverings of S^3 along a non-split alternating link produce a fundamental group which is not left-orderable.

Plans proved in [Pla] that odd branch covering of S^3 along any knot has a double form. That is, there exists G such that

$$H_1(M_k^{(n)}) = G \oplus G \text{ for any odd } k.$$

Turaev produces a related result for k even (compare [Web, DW]). We observe a similar phenomenon in our computation of ABF for $(\sigma_1\sigma_2^{-1})^n$.

Our theorem for the Reduced Fox Coloring Group corresponds well with Plan's and Turaev's results, but what does this mean in the generalization to the ABF module?

Is there some reasonable interpretation of $(\sigma_1\sigma_2^{-1})^\infty$ in relation to ABF module?

6. Acknowledgments

The third author was partially supported by the Simons Collaboration Grant 637794.

References

[A-V] J. W. Alexander, Letter to Oswald Veblen, 1919, Papers of Oswald Veblen, 1881-1960 (bulk 1920-1960), Archival Manuscript Material (Collection), Library of Congress.

[Ale] J. W. Alexander, Topological invariants of knots and links, *Trans. Amer. Math. Soc.* 30 (1928) 275-306.

[BGW] S. Boyer, C. McA. Gordon, L. Watson, On L-spaces and left-orderable fundamental groups, *Math. Ann.* 356 (2013), no. 4, 1213–1245. e-print: arXiv:1406.6718 [math.GT].

[Burau] W. Burau , Über Zopfgruppen und gleichsinnig verdrillte Verkettungen, *Abh. Math. Sem. Univ. Hamburg*, 11: 179-186; 1936.

[Bur-Z] G. Burde, H. Zieschang, Knots. Second edition. De Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 2003. xii+559 pp.

[CF] R. H. Crowell, R. H. Fox, Introduction to knot theory. Reprint of 1963 original, *Graduate Texts in Mathematics*, No 57, Springer-Verlag, New York-Heidelberg, 1977.

[DaPr] M.K. Dabkowski, J.H. Przytycki, A.A Togha, Non-left-orderable 3-manifold groups, *Canad. Math. Bull.* 48 (2005), no. 1, 32–40.

[DW] P. Del Val, C. Weber; Plans' Theorem for Links, *Topology Appl.* 34(1990), no.3, 247-255.

[DJP] J. Dymara, T. Januszkiewicz, J.H. Przytycki, Symplectic structure on colorings, Lagrangian tangles and Tits buildings, *Bulletin of Polish Acad. Sci. Math.*, Vol. 68, No. 2, 2020, 169-194.

[Koshy] T. Koshy, Fibonacci and Lucas numbers with applications, *Pure and applied Mathematics, A Wiley-Interscience Series of Texts, Monographs, and Tracts*, 2001.

[Min] J. Minkus, The branched cyclic coverings of 2 bridge knots and links. *Mem. Amer. Math. Soc.* 35 Nr. 255 (1982), 1–68.

[MV] M. Mulazzani, A. Vesnin, The many faces of cyclic branched coverings of 2-bridge knots and links.(English summary) Dedicated to the memory of Professor M. Pezzana (Italian) *Atti Sem. Mat. Fis. Univ. Modena* 49(2001), 177-215. e-print: https://arxiv.org/abs/math/0106164.

[Mye] B. R. Myers, Number of Spanning Trees in a Wheel, IEEE Transactions on Circuit Theory, CT-18 (1971) 280-282.

[Pla] A. Plans, Contribution to the study of the homology groups of the cyclic ramified coverings corresponding to a knot.(Spanish) Rev. Acad. Ci. Madrid47(1953), 161–193 (5 plates).

[Prz1] J.H. Przytycki, 3-coloring and other elementary invariants of knots. Knot theory (Warsaw, 1995), 275-295, *Banach Center Publ.*, 42, Polish Acad. Sci. Inst. Math., Warsaw, 1998. 57M25 arXiv:math/0608172 [math.GT]

[Prz2] J. H. Przytycki, From Goeritz matrices to quasi-alternating links;
A Book Chapter (Chapter 9) in *The Mathematics of Knots: Theory and Applications*, M.Banagl, D.Vogel, editors; Springer-Verlag, November 30, 2010, pp. 257-315;
e-print: `http://front.math.ucdavis.edu/0909.1118`

[Prz3] J.H. Przytycki, Knots and Graphs: two centuries of interaction; in Knot Theory and Its Applications *Contemporary Mathematics*, Vol. 670, 171-257, September, 2016.

[PBIMW] J.H. Przytycki, Rhea Palak Bakshi, Dionne Ibarra, Gabriel Montoya-Vega, Deborah Weeks, Lectures in Knot Theory; An Exploration of Contemporary Topics
Accepted to Springer Universitexts. To appear Dec. 2023 or Jan. 2024.

[Rol] D. Rolfsen, *Knots and links*. Publish or Perish, 1976. (second edition, 1990; third edition, AMS Chelsea Publishing, 2003).

[Sed] J. Sedlacek, On the Skeletons of a Graph or Digraph, Proceedings of the Calgary International Conference of Combinatorial Structures and Their Applications, Gordon and Breach, 1970, 387-391.

[Web] C. Weber, La demonstration de J. Levine des théorème de A. Plans, Lect. Notes Math. 1051 (1984), pp. 315-330.

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