Borromean Rings: Qual August 2023

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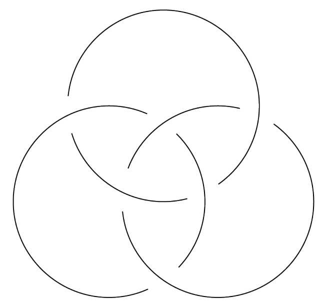
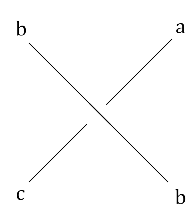


Figure 1: Borromean Rings

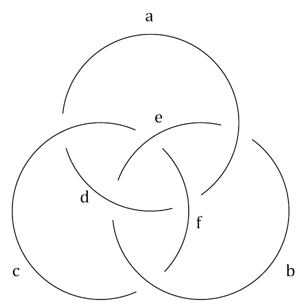
The group $Col(L)$ of the link diagram L is an abelian group which can be computed from any diagram of L as follows:

- The generators are indexed by arcs of the diagram (an arc is a segment of the diagram from tunnel to tunnel)
- Relations are given by the crossings of the diagram and they are of the form $2b - a - c$ where a, b, and c are generators associated to arcs of the crossing as in the figure below



1. Find the group $Col(L)$ for the Borromean Rings.
2. Present the group using only three generators.
3. Can the group $Col(L)$ be presented using only 2 generators?

Proof. 1. We begin by labelling the arcs of the diagram L.



Then the relations are

$$a = 2f - d \tag{1}$$
$$b = 2d - e \tag{2}$$
$$c = 2e - f \tag{3}$$
$$d = 2c - a \tag{4}$$
$$e = 2a - b \tag{5}$$
$$f = 2b - c \tag{6}$$

so

$$Col(L) = \langle a, b, c, d, e, f \mid 2f-d-a = 2d-e-b = 2e-f-c = 2c-a-d = 2a-b-e = 2b-c-f = 0 \rangle.$$

2. We choose a, b, c to be the generators of our next presentation.
Then
from (1),

$$\begin{aligned} a &= 2f - d \\ &= 2(2b - c) - (2c - a) \\ \implies 0 &= 4b - 4c; \end{aligned}$$

from (2),

$$\begin{aligned} b &= 2d - e \\ &= 2(2c - a) - (2a - b) \\ \implies 0 &= 4c - 4a; \end{aligned}$$

from (3)

$$\begin{aligned} c &= 2e - f \\ &= 2(2a - b) - (2b - c) \\ \implies 0 &= 4a - 4b; \end{aligned}$$

from (4)

$$\begin{aligned}
& d = 2c - a \\
\implies & 0 = 2c - a - (2f - a) \\
\implies & 0 = 2c - a - (2(2b - c) - a) \\
\implies & 0 = 4c - 4b; \text{ (same as (1))}
\end{aligned}$$

from (5)

$$\begin{aligned}
& e = 2a - b \\
\implies & 0 = 2a - b - (2d - b) \\
\implies & 0 = 2a - b - (2(2c - a) - b) \\
\implies & 0 = 4a - 4c; \text{ (same as (2))}
\end{aligned}$$

and from (6)

$$\begin{aligned}
& f = 2b - c \\
\implies & 0 = 2b - c - (2e - c) \\
\implies & 0 = 2b - c - (2(2a - b) - c) \\
\implies & 0 = 4b - 4a; \text{ (same as (3))}
\end{aligned}$$

Thus, our presentation with 3 generators is

$$\begin{aligned}
Col(L) &= \langle a, b, c \mid 4b - 4c = 4c - 4a = 4a - 4b = 0 \rangle \\
&= \langle a, b, c \mid 4(b - c) = 4(c - a) = 4(a - b) = 0 \rangle.
\end{aligned}$$

A theorem says that one of the above relations is a consequence of the other and that we can remove it from our presentation. One way to see this is from the fact that

$$-4(c - a) - 4(b - c) = 4(b - a),$$

demonstrating that the last relation is linear combination of the first two, and thus can be removed from the presentation.

Then

$$\begin{aligned}
Col(L) &= \langle a, b, c \mid 4(b - c) = 4(c - a) = 4(a - b) = 0 \rangle \\
&= \langle a, b, c \mid 4(b - c) = 4(c - a) = 0 \rangle
\end{aligned}$$

For more on simplifying group presentations, check here `https://en.wikipedia.org/wiki/Tietze_transformations`

Let $x = b - c, y = c - a$.

Then

$$\begin{aligned}
Col(L) &= \langle a, b, c \mid 4(b - c) = 4(c - a) = 0 \rangle \\
&= \langle a, b, c, x, y \mid 4(x) = 4(y) = 0 \rangle \\
&= \langle c - y, x + c, c, x, y \mid 4x = 4y = 0 \rangle \\
&= \langle c, x, y \mid 4x = 4y = 0 \rangle \\
&= \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4
\end{aligned}$$

Note that the free part of $Col(L)$ represents trivial colorings. That is, the assignment of any word in the free abelian group to all arcs of the diagram.

3. There is no way to present this group with two generators. Each of the three distinct summands will have a generator since $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is not cyclic.

$\square$