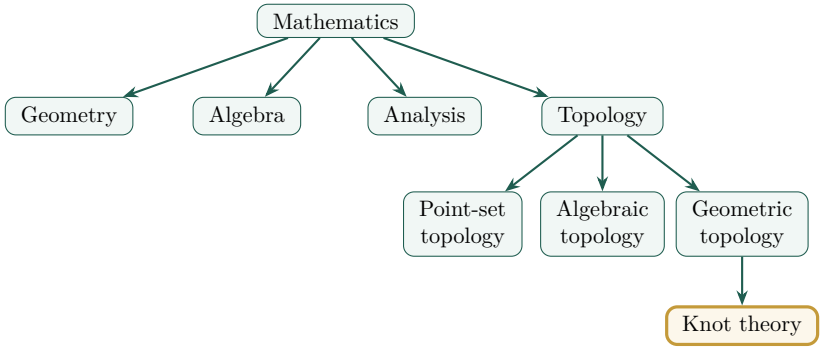
Knot Invariants and Notions of Equivalence

Lecture Notes for the CUA Math Club



Contents

1 (In-)Formal Math **1**

2 What is a knot? **2**

2.1 Historical Motivation 3

3 What does "the same" mean? **3**

3.1 Equivalence relations 4

4 Equivalence of knots **5**

5 Knot diagrams and Reidemeister moves **5**

6 Knot invariants **6**

7 EXTRA: The Hunt For Good Invariants **8**

1 (In-)Formal Math

In the 300+ years since Leibniz and Newton, the way we understand and teach calculus has come a long way. Today, you can walk into any Calc 1 class at any accredited university and you should

expect to see the same set of topics, the same notation, the same explanations, and the same types of practice problems. In three centuries, mathematicians reached a consensus on what calculus *is* and the "best way" to teach it. And this formalization is probably a good thing for students and scientists overall; we have a shared language and access to lots of legible, high-quality resources.

What I want to do today, though, is give you a sense of how *informal* and messy math research can be—decidedly unlike polished calculus we find in our textbooks and classrooms.

When mathematicians set out to study a new object, like we will today, they can spend an enormous amount of time just searching for the right formalization—the right "language". In the case of calculus, Newton and Leibniz's ideas were only rigorously formalized in the early 19th century (Cauchy, Weierstrass, etc.) and the language they used was very different ("fluxions").

Hopefully today's excercise will capture some of the messiness involved in trying to formally study a new mathematical object: knots!

We will try and create the mathematical formalism to answer the question *how can we tell if two knots are different?*

The second question leads us to equivalence relations, knot diagrams, Reidemeister moves, and knot invariants.

2 What is a knot?

Informally, a knot is a loop of string sitting in three-dimensional space. The loop may be stretched all you like, but no cutting is allowed. Imagine an infinitely stretchy, knotted-up rubber band.

More precisely, we have the following definition.

Definition 2.1 (Knot)**.** A *knot* is a smooth embedding[1]

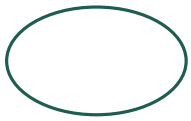
$$K\colon S^1 \hookrightarrow \mathbb{R}^3.$$

Activity—Make a knot

Use a length of string to create a closed loop. You may tie the string however you wish before joining its ends.
Rules. Once the ends are joined, you may stretch or move the string, but you may not cut it, glue it, or pass one strand through another.

The simplest example is the *unknot*, a circle. A familiar nontrivial example is the *trefoil knot*.

[1]The word *embedding* packages two conditions: the map is continuous and one-to-one, and it identifies the circle with its image in a well-behaved way. For this discussion, it is enough to picture an infinitely thin, infinitely stretchy loop of string.



unknot



trefoil

2.1 Historical Motivation

As you work on your constructing your knots, I wanted to tell you why mathematicians first became interested in formalizing the mathematics of knots.

In the 1860s, scientists were wondering what atoms might look like. And Scottish scientist Peter Guthrie Tait was excited by a new theory proposed by his friend Lord Kelvin. Kelvin believed that the atoms which comprise the world were, in fact, knots. He called them “knotted vortices" which he believed "floated in the luminiferous aether."

Tait had already been conducting experiments on smoke rings, which demonstrated some knotting behavior, so, inspired by Kelvin, Tait set about on a very reasonable path of scientific inquiry: he tried to construct a *periodic table of knots*.

Obviously, we know today that atoms are not knots, but it’s funny to imagine a world where Tait’s knot table appeared on the walls of science classrooms around the world.

Activity—Brainstorm a Periodic Table of Knots

How would we arrange the knots in this classroom into a periodic table? What initial questions should we answer?

- Are there infinitely many?
- How can we tell them apart?
- Can we turn them into mathematical symbols/diagrams?
- How should we stratify and organize different knots in the table?

3 What does “the same” mean?

In fact, the central question of knot theory, which has kept mathematicians busy since Tait, is simply stated: **How can we tell two knots apart?** To answer this question, we first need a notion of same/different.

Are the following pairs of mathematical objects the same?

- The integers 1 and 13.
- Two non-equal, similar triangles.
- A path graph on three vertices and a “bent” path graph on three vertices.

- A coffee mug and a doughnut.
- The fractions $\frac{1}{2}$ and $\frac{32}{64}$.

The answer: *it depends*. We first have to decide which features matter in context.

The numbers 1 and 13 are unequal as integers, but they are congruent modulo 12. Two triangles of different sizes are not congruent, but we might only care about their similarity. The two fractions are written differently but represent the same rational number.

Similarly, a graph isomorphism ignores the placement of a graph on the page and remembers only which vertices are adjacent. Topology ignores measurements such as lengths and angles, and studies what survives "continuous deformation."

Key idea

Mathematical objects are not simply "the same" or "different." They are the same or different *with respect to a chosen notion of equivalence.*

3.1 Equivalence relations

Definition 3.1 (Equivalence relation)**.** Let X be a set. A relation $\sim$ on X is an *equivalence relation* if, for all $x, y, z \in X$, it is

(i) *reflexive*: $x \sim x$;

(ii) *symmetric*: if $x \sim y$, then $y \sim x$;

(iii) *transitive*: if $x \sim y$ and $y \sim z$, then $x \sim z$.

An equivalence relation partitions a set into *equivalence classes*. Each class collects all objects that we have agreed to regard as the same. Different areas of mathematics can be viewed, in part, as the study of sets equipped with different notions of equivalence.

Area	Objects	Typical notion of equivalence
Number theory	integers	equality/congruence modulo n
Geometry	figures	congruence/similarity
Graph theory	graphs	graph isomorphism
Topology	spaces	homeomorphism/homotopy/isotopy
Knot theory	knots	ambient isotopy

The traditional joke is that a topologist cannot distinguish a coffee mug from a doughnut: each has one hole, and one can be continuously deformed into the other. The joke is really about choosing homeomorphism, rather than congruence, as the relevant equivalence relation.

4 Equivalence of knots

For knots, the right notion of equivalence is called *ambient isotopy*. The word *ambient* emphasizes that the entire surrounding space moves along with the knot, preventing one strand from passing through another.

Definition 4.1 (Equivalent knots)**.** Two knots $K_0, K_1 \subseteq \mathbb{R}^3$ are *equivalent* if there is a continuous family of homeomorphisms

$$H_t \colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad 0 \leq t \leq 1,$$

such that H_0 is the identity and $H_1(K_0) = K_1$. Such a family is called an *ambient isotopy*.

In the string model, ambient isotopy means that we may stretch and move the string without cutting it or letting it pass through itself.

Activity—Compare with a neighbor

Working with a neighbor, decide whether your two string knots are equivalent. You may manipulate them using the rules above. Be ready to defend your answer.
Discussion questions.

- What evidence convinces you that two knots are the same?
- What evidence convinces you that they are different?
- Is it easier to prove sameness or difference?

This last question exposes an asymmetry. To show that two knots are the same, it is enough to exhibit one deformation from one to the other. To show that they are different, we must somehow rule out *every possible* deformation.

5 Knot diagrams and Reidemeister moves

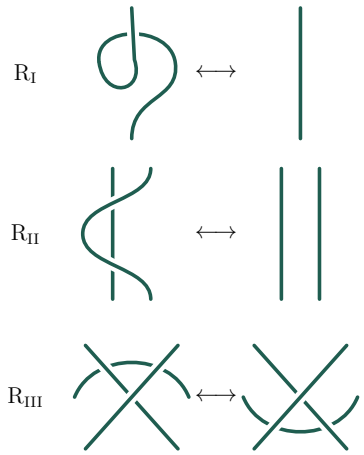
Ambient isotopy is a really complicated definition to work with, so let's find a simpler description of knot equivalence.

Let's start by projecting our knots into a plane. At every crossing, we interrupt the lower strand to record which strand lies on top. The resulting picture is a *knot diagram*. A single knot has ∞-many different diagrams (because we can stretch then project).

The following three local changes are the *Reidemeister moves*:

Theorem 5.1 (Reidemeister's theorem)**.** *Two knot diagrams represent equivalent knots if and only if one diagram can be transformed into the other by a finite sequence of Reidemeister moves together with planar deformations of the diagram.*

Reidemeister's theorem translates a three-dimensional equivalence problem into a two-dimensional, combinatorial one. It captures the geometric notion of ambient isotopy in the language of diagrams.



All rotated and reflected versions are understood.

Activity—Reidemeister Moves

- Lay your knot flat on the table and draw a diagram for your knot.
- Perform one of each reidemeister move, first on your physical knot, then on your diagram.
- Bonus: can you decrease the number of crossings your knot has until no more reductions are possible?

6 Knot invariants

So now we have developed a simpler description of when two knots are the same using planar diagrams, but it would still be difficult to fill out our periodic table since:

Key idea

If two knots are the different, it means that no possible combination of Reidemeister moves can transform one diagram into the other.

But there are infinitely many applications of Reidemeister moves possible!

The most useful strategy for proving that two objects are different is to assign to each object a quantity that does not change under the chosen equivalence.

Definition 6.1 (Knot invariant)**.** A *knot invariant* is a function I from knots to some set of values such that

$$K_0 \sim K_1 \quad \Longrightarrow \quad I(K_0) = I(K_1).$$

In other words, equivalent knots always receive the same value.

The contrapositive is the useful part:

$$I(K_0) \neq I(K_1) \quad \Longrightarrow \quad K_0 \not\sim K_1.$$

Thus a knot invariant can prove that two knots are different. The converse need not hold: two different knots may have the same value of a particular invariant.

Key idea

To fill out our periodic table of knots, we should

1. come up with some knot invariants,
2. compute the invariants for a collection of knots,
3. start separating the knots which have different values on some invariant.

Example 6.2 (Unknotting number)**.** The *unknotting number* $u(K)$ is the minimum number of crossing changes needed to turn a diagram of K into a diagram of the unknot, where the minimum is taken over all diagrams of K. It satisfies

$$u(\text{unknot}) = 0,$$

while the trefoil has unknotting number 1.

Other invariants include the crossing number, colorability tests, the knot group, and knot polynomials. Each remembers some information while forgetting other information. The search for stronger and more computable invariants remains a central theme in knot theory.

Key idea—The classification strategy

1. Specify the objects: knots are embedded circles in three-dimensional space.
2. Specify equivalence: knots are considered the same up to ambient isotopy (Reidemeister moves).
3. Encode the objects: use two-dimensional knot diagrams.
4. Encode equivalence: use Reidemeister moves.
5. Distinguish equivalence classes: compute knot invariants.

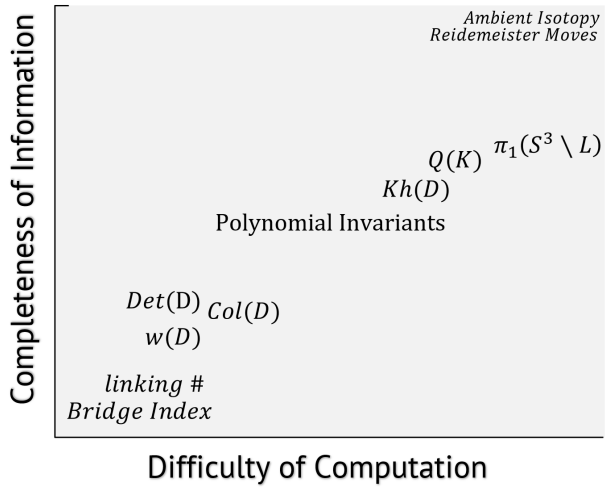
Questions for further discussion

1. Can any finite list of invariants distinguish every pair of knots?
2. How might one systematically list knots without counting the same knot many times?
3. Which properties of a diagram depend on the drawing, and which are genuine properties of the knot?
4. What should count as an efficient algorithm for recognizing the unknot?

THANK YOU!

7 EXTRA: The Hunt For Good Invariants

There's something of an art to finding good invariants because of the following graph



Definition 7.1 (Fundamental quandle)**.** Let D be an oriented diagram of a knot K, with one generator x_i for each arc of D. At a crossing, write x and z for the incoming and outgoing under-arcs, y for the over-arc, and $\varepsilon \in \{+1, -1\}$ for the sign of the crossing. The *fundamental quandle* of K is

$$Q(K) = \langle x_1, \ldots, x_n \mid z = x \triangleright^{\varepsilon} y \text{ at every crossing} \rangle_{\mathbf{Qnd}},$$

where

$$x \triangleright^{+1} y = x \triangleright y, \qquad (x \triangleright^{-1} y) \triangleright y = x.$$

Its isomorphism class is independent of the chosen diagram.

Theorem 7.2 (Joyce–Matveev)**.** *For oriented classical knots,*

$$Q(K) \cong Q(K') \quad \Longleftrightarrow \quad K' \simeq K \;\; \textit{or} \;\; K' \simeq -K^*,$$

where K^* is the mirror of K and $-K$ denotes orientation reversal. Thus $Q(K)$ is a complete invariant up to reverse-mirroring.

The Kauffman bracket Let D be an unoriented link diagram. A *state* s chooses an A- or B-smoothing at every crossing. Write

$$a(s) = \#\{A\text{-smoothings}\}, \qquad b(s) = \#\{B\text{-smoothings}\},$$

and let $|s|$ be the number of circles in the resulting state.

Setting

$$\delta = -A^2 - A^{-2},$$

the *Kauffman bracket* is

$$\boxed{\langle D \rangle = \sum_s A^{a(s)-b(s)} \delta^{\,|s|-1}} \in \mathbb{Z}[A, A^{-1}].$$

Equivalently, it is characterized by

$$\langle \bigcirc \rangle = 1, \qquad \langle D \sqcup \bigcirc \rangle = \delta \langle D \rangle ,$$

together with the local smoothing relation

$$\langle \text{crossing} \rangle = A \langle A\text{-smoothing} \rangle + A^{-1} \langle B\text{-smoothing} \rangle .$$