THE GEORGE WASHINGTON UNIVERSITY

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Gram Determinants of Type Mb & Representations of the Temperley-Lieb Algebra

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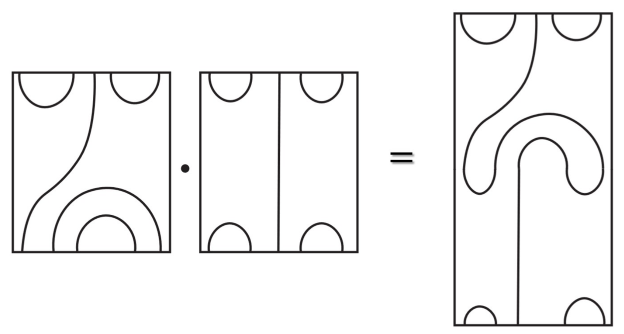
The Temperley-Lieb Algebra

The Temperley-Lieb algebra Tl_n

Let R be a commutative ring and fix $\delta \in R$. The Temperley–Lieb algebra $Tl_n(\delta)$ is the unital associative R-algebra generated by the elements $e_1, e_2, \ldots, e_{n-1}$, subject to the Jones relations:

- $e_i^2 = \delta e_i$ for all $1 \le i \le n-1$,
- $e_i e_{i+1} e_i = e_i$ for all $1 \le i \le n-2$,
- $e_i e_{i-1} e_i = e_i$ for all $2 \le i \le n-1$,
- $e_i e_j = e_j e_i$ for all $1 \le i, j \le n-1$ such that $|i-j| \ne 1$.

Multiplication:



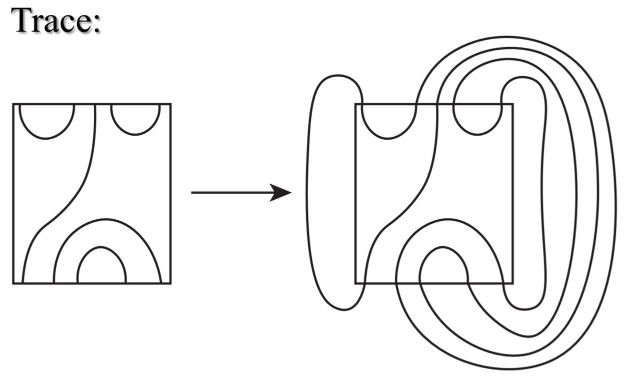
The trace function on Tl_n defines a natural bilinear form.

For $x, y \in Tl_n$, define

$$\langle x, y\rangle = tr(x\bar{y}) \in \mathbb{Z}[\delta]$$

where $\bar{y}$ is the vertical reflection of the diagram y.

Trace:



The matrix $G_n = \{\langle x, y\rangle\}_{x,y \in Tl_n}$ is called the *Gram Matrix*

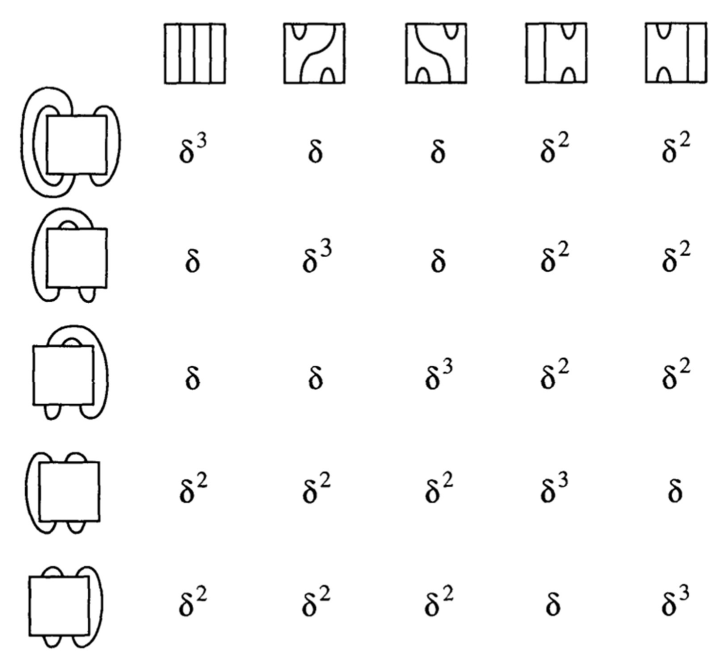
The determinant $\det(G_n)$ is called the *Gram Determinant*.

Gram Determinant of Type A

Theorem

$$\det(G_n) = \Delta_1^{c_n} \prod_{k=1}^{n} \left(\frac{\Delta_k}{\Delta_{k-1}}\right)^{\alpha_k}$$

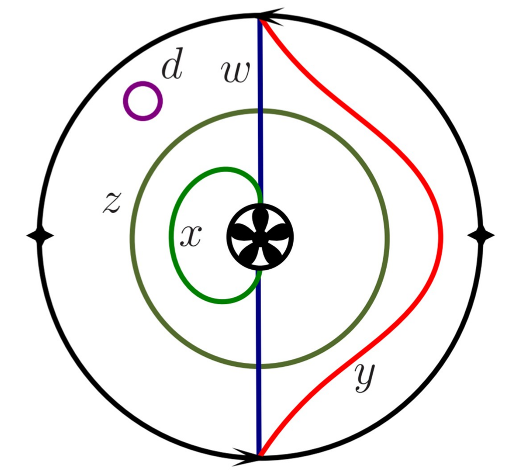
where $\Delta_i = \frac{(-1)^i (A^{2(i+1)} - A^{-2(i+1)})}{A^2 - A^{-2}}$, $c_n = \frac{1}{n+1}\binom{2n}{n}$, *and* $\alpha_k = \binom{2n}{n-k} - \binom{2n}{n-k-1}$



Identifying Curves in Kb

$$\langle\, , \,\rangle : Mb_{2n,n}^{\oplus 2} \to \mathbb{Z}[d, z, x, y, w]$$

is defined by composing Mb crossingless connections and identifying the homotopy classes of the curves in Kb.

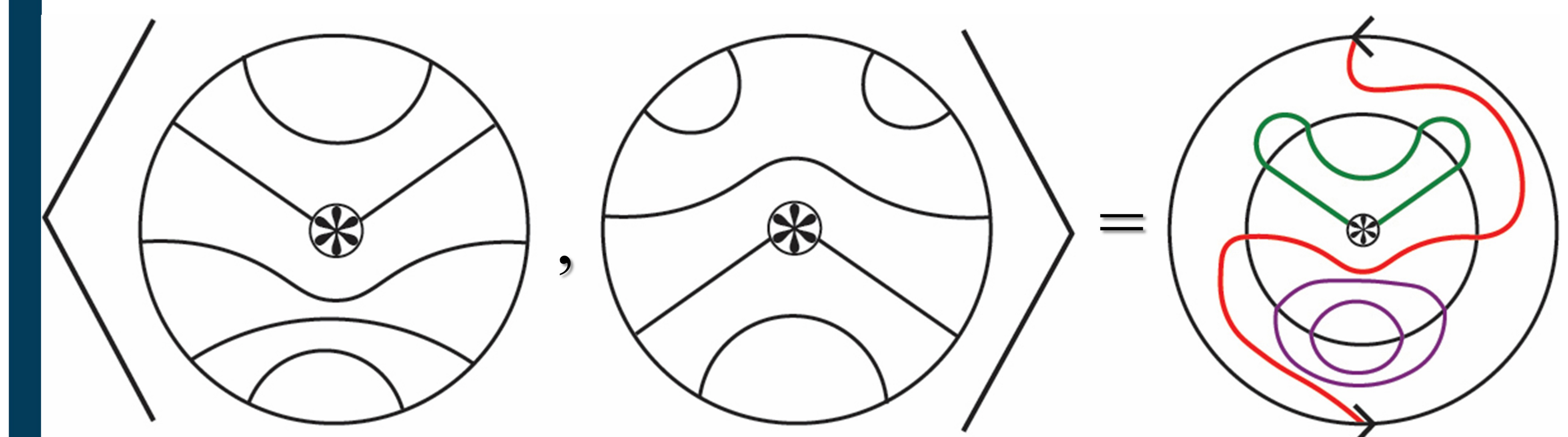


The five homotopy classes of simple closed curves in Kb.

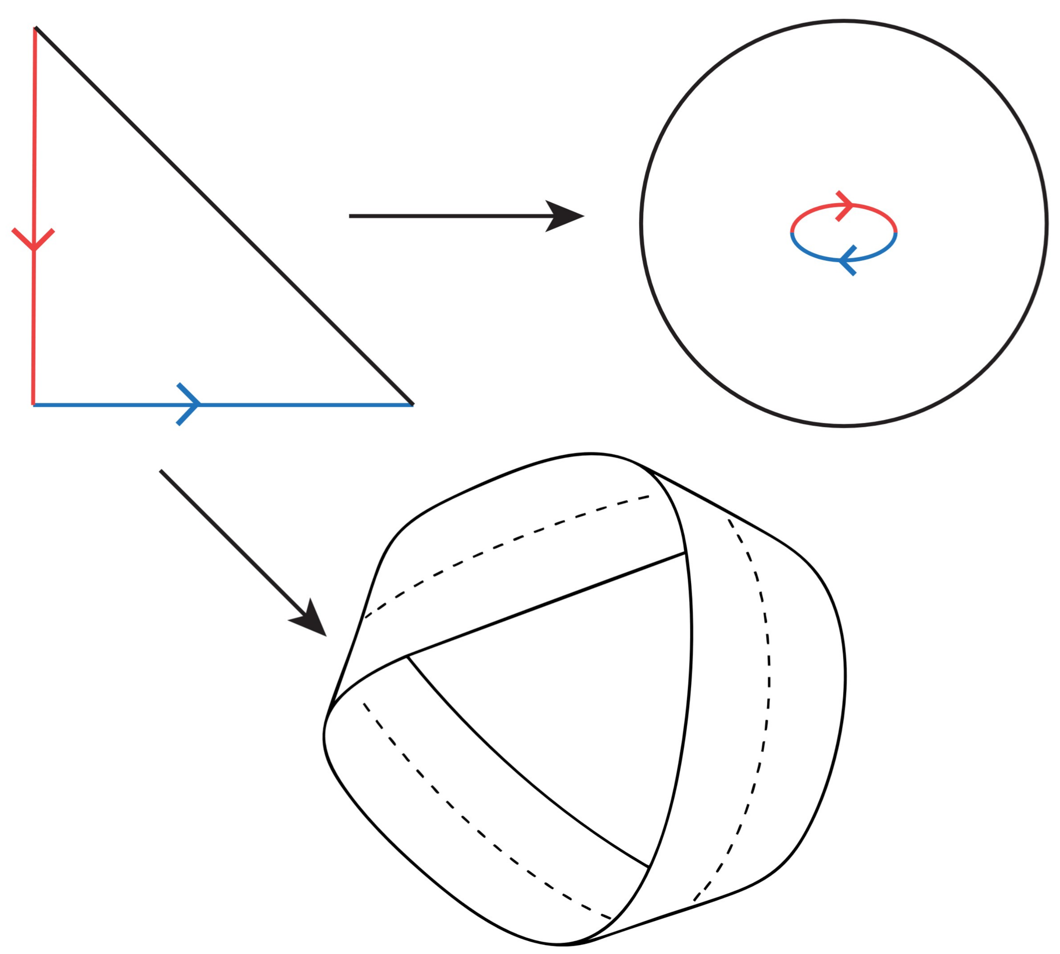
Bilinear form on Crossingless Connections in the Mobius Band

Q: What is the Gram Determinant for the Bilinear Form of Mb crossingless connections?

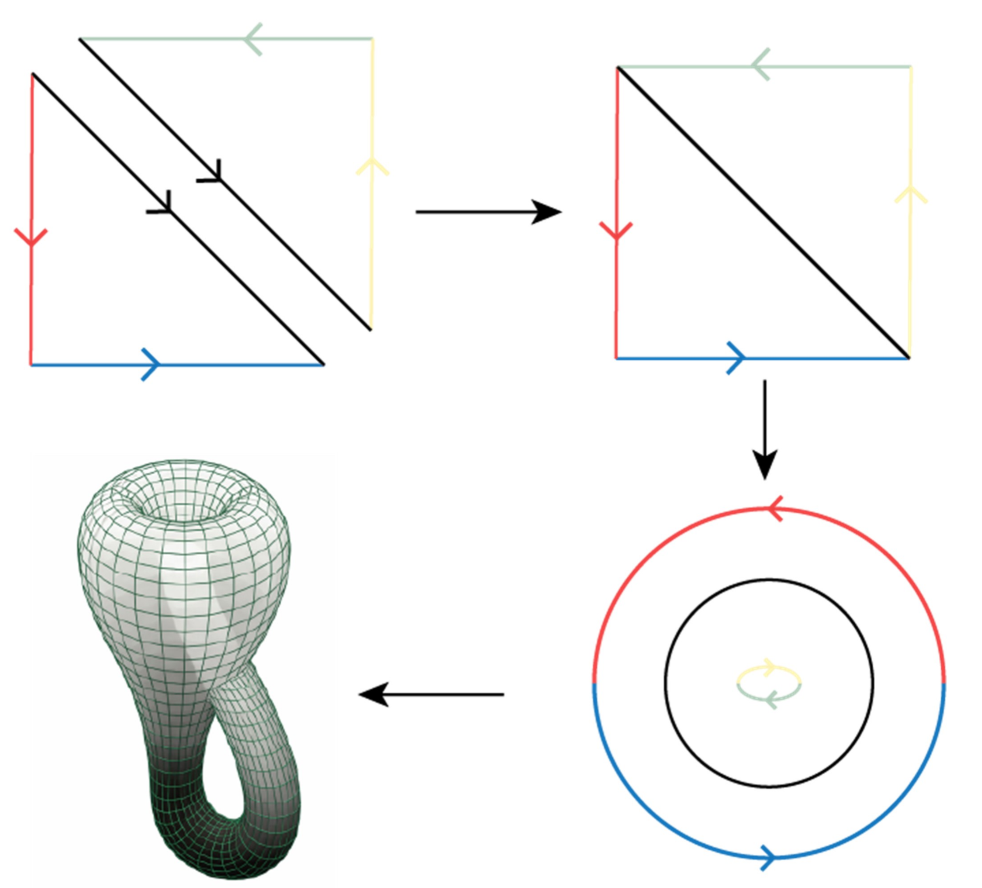
$$\det(G_{2n,n}^{Mb}) := D_n^{Mb}$$



The Mobius Band



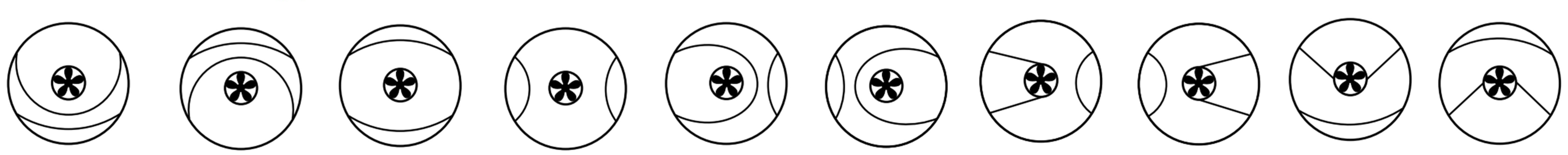
The Klein Bottle



Crossingless Connections in $Mb_{4,2}$

Theorem

$$|Mb_{2n,n}| = \sum_{k=0}^{n} \binom{2n}{k}$$



Results and Conjectures

Conjecture

$$D_n^{Mb}(d, w, x, y, z) = \prod_{k=1}^{n} (T_k(d) + (-1)^k z)^{\binom{2n}{n-k}}$$
$$\prod_{\substack{k=1 \\ k \text{ odd}}}^{n} ((T_k(d) - (-1)^k z) T_k(w) - 2xy)^{\binom{2n}{n-k}}$$
$$\prod_{\substack{k=1 \\ k \text{ even}}}^{n} ((T_k(d) - (-1)^k z) T_k(w) - 2(2-z))^{\binom{2n}{n-k}}$$
$$\prod_{i=1}^{n} D_{n,i},$$

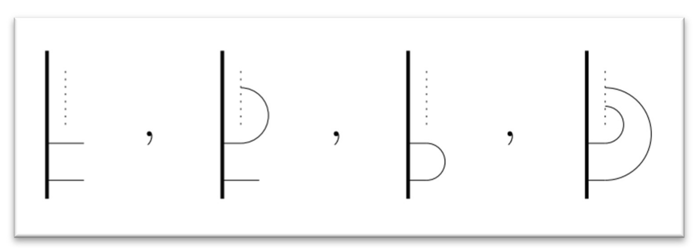
where $D_{n,i} = \prod_{k=1+i}^{n} (T_{2k}(d) - 2)^{\binom{2n}{n-k}}$ and i represents the number of curves passing through the crosscap.

$$G_{6,3}^{\tilde{Mb}} = \begin{pmatrix}
d^2w & 0 & 0 & dw & 0 & 0 & dw & dw & 0 & w & w & w & dw & dw & 0 \\
0 & d^2w & 0 & 0 & dw & dw & dw & dw & 0 & w & w & w & 0 & 0 & dw \\
0 & 0 & d^2w & 0 & dw & dw & 0 & 0 & dw & w & w & w & dw & dw & 0 \\
dw & 0 & 0 & d^2w & 0 & 0 & w & w & w & dw & dw & 0 & w & w & w \\
0 & dw & dw & 0 & d^2w & 0 & w & w & w & dw & dw & 0 & w & w & w \\
0 & dw & dw & 0 & 0 & d^2w & w & w & w & 0 & 0 & dw & w & w & w \\
dw & dw & 0 & w & w & w & d^2w & 0 & 0 & dw & 0 & dw & w & w & w \\
dw & dw & 0 & w & w & w & 0 & d^2w & 0 & 0 & dw & 0 & w & w & w \\
0 & 0 & dw & w & w & w & 0 & 0 & d^2w & dw & 0 & dw & w & w & w \\
w & w & w & dw & dw & 0 & dw & 0 & dw & d^2w & 0 & 0 & dw & 0 & 0 \\
w & w & w & dw & dw & 0 & 0 & dw & 0 & 0 & d^2w & 0 & 0 & dw & dw \\
w & w & w & 0 & 0 & dw & dw & 0 & dw & 0 & 0 & d^2w & 0 & dw & dw \\
dw & 0 & dw & w & w & w & w & w & w & dw & 0 & 0 & d^2w & 0 & 0 \\
dw & 0 & dw & w & w & w & w & w & w & 0 & dw & dw & 0 & d^2w & 0 \\
0 & dw & 0 & w & w & w & w & w & w & 0 & dw & dw & 0 & 0 & d^2w
\end{pmatrix}$$

Theorem

$$D_n^{Mb} = \det(\tilde{G}^{Mb_{n,1}}) = d^{(n-1)c_{n-1}} \det(\tilde{G}^{Mb_{n-1,1}}) \det(G_{n-1}^A) \det(D')$$

$$\tilde{G}^{Mb_{n,1}} = \begin{pmatrix} A & B^T \\ B & D \end{pmatrix}$$



$$A = \begin{pmatrix} d\tilde{G}^{Mb_{n-1,1}} & 0 \\ 0 & G_{n-1}^A \end{pmatrix}$$

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