# *Research Statement*

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## Research Overview

Knot theory studies smooth embeddings of circles in 3-dimensional space up to isotopy. The deceptively simple central question—how can we distinguish knots and links?—has precipitated the development of a rich theory which reaches for tools in algebraic topology, representation theory, combinatorics, topological quantum field theory, and category theory.

Knot invariants are functions which are constant on isotopic knots, though the converse need not be true. Knot invariants enjoy a natural tradeoff between (1) the completeness of the information they provide and (2) their computability. Famous knot invariants like the Jones polynomial manage this tradeoff effectively.

My work is mainly concerned with the algebraic and geometric properties of knots and related objects. A common question throughout my work is how local relations produce global information. In a skein module, a local relation among link diagrams determines an algebraic invariant of a 3-manifold. In a diagrammatic algebra, connecting and closing diagrams produces a bilinear form whose determinant records representation-theoretic information. In nonassociative algebra, relations among bracketed products produce boundary maps and homology theories.

I develop these ideas through three related programs. The first studies cubic skein modules and their connections to braid algebras. The second studies Gram determinants of diagrammatic algebras, particularly those arising from crossingless connections in the Möbius band. The third constructs and computes homology theories for quasigroups and Yang–Baxter operators. A recent result is my proof of the conjectured Gram determinant for Möbius connections containing one arc through the crosscap, using the representation theory of the periodic Temperley–Lieb algebra. These projects have also produced finite reduction algorithms for cubic skein modules, Chebyshev identities for Möbius-band Gram determinants, and new low-dimensional homology theories for Bol–Moufang quasigroups.

## Skein Modules

Skein modules are 3-manifold invariants similar in spirit to homotopy and homology groups, and were first introduced by my advisor Jozef Przytycki and, independently, by Vladimir Turaev. Given a 3-manifold $M$ and a commutative ring $R$, consider the $R$-module spanned by all ambient isotopy classes of links embedded in $M$. A skein relation—which relates links differing only in a small neighborhood—is chosen, and the module of link classes is quotiented by the submodule generated by that relation. Intuitively, the skein relation provides a means of "breaking crossings" in embedded links. The main task in the study of skein modules is to compute their algebraic structure. The difficulty of this task differs wildly based on the choice of manifold, skein relation, and ring.

My work has focused on cubic skein modules. The cubic skein relation has the form




where the tangles are understood as local pictures of links which agree outside a 3-ball.

Even with the simplest choice of 3-manifold, $S^3$, a complete structure theorem for cubic skein modules remains unknown. The difficulty comes from the fact that not every link is 3-move reducible to a trivial link, while the cubic skein relation is a deformation of the 3-move. A complete description of equivalence classes under 3-moves is also unknown. Thus, even basic questions about generators, relations, and dimension remain open.

In my work, I developed and implemented algorithms to compute broad families of links in terms of unknots. For rational links, the algorithm uses continued fraction presentations to reduce one twist region at a time. For pretzel links, it reduces each column to a small collection of basic tangles. The data collected guided the search for ideals in the cubic skein module, and the resulting formulas describe the relations produced by these families. In joint work, we also proved that every 3-algebraic tangle can be reduced to a linear combination of forty basic tangles (`arXiv:2511.10959`). This gives a finite setting in which cubic skein calculations can be carried out exactly.

I plan to extend this work to other families of links, recovering more of the structure of the cubic skein module of $S^3$. I also plan to leverage the theory of cubic Hecke algebras to prove dimension results and further compute ideals. The larger goal is to determine whether a genuinely cubic specialization can produce a new polynomial invariant of knots rather than a specialization of an existing quadratic invariant. Other concrete questions include whether cubic skein modules detect Fox 7-colorings or wrapping number in the solid torus.

## Gram Determinants in Diagrammatic Topology

### Möbius-band and annular structures

The Temperley–Lieb algebra is equipped with a natural bilinear form in which two crossingless connections are first connected and then closed, producing a finite number of unknots. This bilinear form was studied by Lickorish in his skein-theoretic construction of the Witten–Reshetikhin–Turaev invariants. The determinant of its Gram matrix detects exactly when the form degenerates, and its factorization reflects the representation theory of the algebra.

Extending this idea, we can consider an analogous bilinear form on crossingless connections between $2n$ marked boundary points in the Möbius band. These elements form a basis for the relative Kauffman bracket skein module of the Möbius band. The bilinear form is defined by identifying two Möbius bands along their boundary. The resulting surface is a Klein bottle, and the closed curves produced by gluing are counted according to their homotopy type. The eventual goal is to produce a closed formula for the Gram determinant on the entire collection of Möbius connections.

In joint work with Dionne Ibarra and Gabriel Montoya-Vega, I developed an algorithm to construct these Gram matrices and determine the homotopy type of each closed component. We proved a predicted family of factors in the determinant and found new identities relating the first and second kinds of Chebyshev polynomials (`arXiv:2504.16439`).

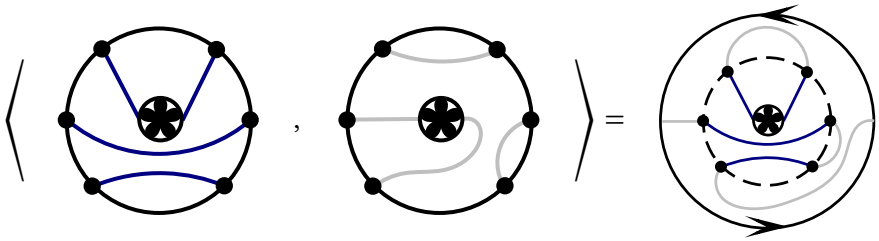


**Figure 1:** A pairing between two crossingless connections in the Möbius Band and the resulting closed curves in the Klein Bottle.

These calculations provided the evidence and algebraic patterns needed for a representation-theoretic proof.

In recent work, I proved the conjectured Gram determinant for the subcollection containing one arc through the crosscap. The proof is representation-theoretic: I use the periodic Temperley–Lieb algebra to construct an alternative basis for the module of Möbius connections. In this basis, the bilinear form reflects the module decomposition, explaining the determinant factors without requiring the expansion of a large symbolic matrix.

Extending this proof to all Möbius connections requires an algebraic structure capable of distinguishing the different homotopy classes of curves in the Klein bottle.

**Gram determinants as universal objects**

Gram determinants in diagrammatic algebra often first appear as experimental factorizations. Representation theory can explain these formulas by providing bases adapted to the decomposition of the underlying algebra or module. For the Hecke algebra of type $A$ and the Temperley–Lieb algebra, cellular and seminormal methods provide such bases in a general framework developed by Mathas. My Möbius-band work applies the same principle: an appropriate change of basis makes the determinant factors visible without requiring the expansion of a large symbolic matrix.

For diagrammatic spaces that carry an algebra structure, there is also a useful universal viewpoint based on multiplication. The one-crosscap Möbius connections above are instead treated as a module; the following construction applies to an algebra and does not assume a multiplication on those connections. Let $A$ be a finite free algebra over a commutative ring $R$, with basis $e_1, \dots, e_N$ and multiplication

$$e_i e_j = \sum_{k=1}^{N} c_{ij}^{k} e_k.$$

The parastrophic matrix of $A$ is

$$P_A(\mathbf{x}) = \left( \sum_{k=1}^{N} c_{ij}^{k} x_k \right)_{i,j}.$$

For any linear functional $\lambda : A \to R$, substituting $x_k = \lambda(e_k)$ gives the matrix with entries $\lambda(e_i e_j)$. For a diagrammatic algebra equipped with an $R$-linear anti-involution $*$, a closure pairing has the form

$$\langle a, b \rangle_\lambda = \lambda(a^* b).$$

If $*$ permutes the chosen diagram basis up to sign, the Gram matrix is obtained from a specialization of $P_A(\mathbf{x})$ by a signed row permutation, so the determinants agree up to sign. For a fixed algebra and such an anti-involution, the parastrophic determinant therefore packages the Gram determinants obtained by varying the closure functional.

I plan to apply this viewpoint to diagrammatic algebras with a natural multiplication, including the finite tangle algebras arising from cubic skein modules. In the Möbius-band setting, the module over the periodic Temperley–Lieb algebra provides a complementary explanation for the one-crosscap determinant factors, without identifying that Gram matrix with a specialization of a parastrophic matrix. The broader aim is to distinguish factors forced by an algebra or module from those introduced by a particular topological closure.

## Homology of Bol–Moufang Quasigroups

Quasigroups are nonassociative magmas with an invertible operation. Said another way, quasigroups have multiplication tables given by Latin squares. Of interest to knot theorists is the close connection between quasigroups and quandles. Quandles are algebras whose axioms encode the Reidemeister moves—isotopy-preserving transformations of link diagrams. Quandles and quasigroups overlap in a class of algebras called Latin quandles.

My work develops a homology theory for a class of quasigroups known as Bol–Moufang quasigroups and uses techniques from the theory of quandle homology. Bol–Moufang quasigroups are quasigroups with one of sixty additional defining identities, each said to "approximate associativity." By studying extensions by affine quasigroups, we obtained boundary maps $\partial_2$ and $\partial_3$ and used them to define the first and second homology groups (`arXiv:2508.21268`).

Using the classification of Phillips and Vojtechovsky, I proved that the resulting homology theory is an invariant of the quasigroup variety rather than of the particular Bol–Moufang identity used to present it. Thus, two identities defining the same variety yield the same homology theory. In the associative case, the construction recovers the standard homology of semigroups.

My proof is geometric in spirit. Each Bol–Moufang identity is represented by a pair of rooted binary trees whose leaves are labeled by quasigroup elements, and the boundary maps determine invariants of these labeled trees. I showed that the algebraic operations relating equivalent Bol–Moufang identities induce corresponding transformations of the associated tree invariants.

The next problem is to construct higher boundary maps and determine whether the homology theories associated to equivalent Bol–Moufang identities continue to agree in higher dimensions. These higher homology groups are not yet defined for every identity or variety. However, preliminary calculations with syzygies among Bol–Moufang rela-

tions suggest a systematic approach to producing more general boundary-map formulas. Once the higher-dimensional theory is established, I plan to compare it with existing homology theories for quasigroups and loops, as well as small-category and quandle homology. Latin quandles provide a particularly natural test case: because they are both quasigroups and quandles, their homology may clarify the relationship between the new theory and quandle homology.

**Related work in Yang–Baxter homology**

I also study the homology of Yang–Baxter operators which produce the HOMFLYPT polynomial. In joint work, we reduced the computation of the $n$th homology of the operator $R_{(m)}$, for arbitrary $m$, to the computation of $n + 1$ initial cases. This reduction allowed us to compute explicit formulas for the third and fourth homology groups (`arXiv:2502.20659`). I plan to prove the rank and torsion patterns suggested by our low-dimensional computations. A long-term goal is to compare Yang–Baxter homology with Khovanov-type homology, possibly through a geometric realization of the Yang–Baxter chain complex.

## Future Directions

My next priority is to extend the one-crosscap Gram determinant result to all Möbius connections. This problem combines methods of computation and representation theory developed in my recent work and asks how much of the determinant is forced by the underlying module structure.

A second direction is to use the finite tangle reductions to study the structure of cubic skein modules. Here, the universal determinant viewpoint may connect the multiplication in tangle algebras to the bilinear forms produced by topological closure.

Alongside these projects, I plan to develop the higher-dimensional homology of Bol–Moufang quasigroups and pursue comparison maps with existing theories. Together, these directions continue the central theme of my research: understanding how local topological and algebraic relations determine global invariants.

## Research with Undergraduate Students

Each of these programs has concrete entry points for undergraduate students. Students with a background in linear algebra or programming can generate small Möbius-band Gram matrices, classify the curves created by gluing, and factor the resulting polynomials. Students can also implement cubic skein reductions for rational or pretzel links and compare the results obtained from different presentations of the same link. In nonassociative algebra, students can begin with Latin squares, test Bol–Moufang identities, construct the low- dimensional boundary matrices, and compute their Smith normal forms. For students earlier in the curriculum, my work on secure domination in Mycielskian graphs and on psyquandle coloring quivers provides projects accessible after an introductory graph theory course. Depending on the student's preparation and the length of the project, the outcome could be reproducible code, a verified data set, a new conjecture, or a theorem suitable for an undergraduate thesis.